Logic and Modelling

— Introduction to Predicate Logic —

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Predicate Logic

In propositional logic there are:

- propositional variables $p, q, r, \ldots$ that can be T or F

In predicate logic there are:

- variables $a, b, \ldots$ that represent **objects** (or **individuals**),
- **predicates** $P(x)$ or **relations** $R(x, y)$ on the objects

1-ary predicates like $P(\_)$ can express properties of objects:

- e.g. $P(x)$ can express that 'x is green'

2-ary predicates like $R(\_, \_)$ can express relations:

- e.g. $R(x, y)$ can express 'x knows y'.

Name natural examples of 3-ary predicates?

- $L(x, y, z) = x, y, z$ are points on same line in the plane

Formulas and Logic Connectives

In predicate logic, the role of propositional variables is taken over by the **atomic formulas** with object/predicate-structure:

The **atomic formulas** are predicates over objects:

- $P(x)$
- $R(x, y)$

Logic connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ keep their role.

We can build **formulas** using propositional connectives, starting from the smallest building blocks of atomic formulas:

- $P(x)$ *(x is green)*
- $P(x) \wedge R(x, y)$ *(x is green and x knows y)*
- $R(x, y) \rightarrow \neg R(y, x)$ *(if x knows y then y does not know x)*

Predicate logic is more expressive

In propositional logic, we could only state p, $p \wedge q$, $p \rightarrow \neg q$.

Quantifiers

Instead of 'x is green' we now want to say:

- 'somebody is green', or
- 'everybody is green'.

Existential quantifier

$$\exists x\ \phi$$

means: there exists some x such that the formula ϕ is true.

$$\exists x\ P(x) \qquad \textit{(somebody is green)}$$

Universal quantifier

$$\forall x\ \phi$$

means: for every x the formula ϕ is true.

$$\forall x\ P(x) \qquad \textit{(everybody is green)}$$

Examples

Examples of formulas

- $\neg \forall x \; P(x)$
  Not everybody is green.
- $\exists x \; \neg P(x)$
  There exists someone who is not green.
- $\forall x \; \neg R(x, x)$
  Everybody does not know himself.
- $\forall x \; (R(x, x) \rightarrow \neg P(x))$
  Everybody, who knows himself, is not green.
- $\forall x \; \forall y \; (R(x, y) \rightarrow \neg R(y, x))$
  For all x and y, if x knows y then y does not know x.

What do these formulas mean given that

- $P(x)$ means 'x is green'
- $R(x, y)$ means 'x knows y'

Not Forall and Not Exists

Let us reconsider two examples

$\neg\forall x\ P(x)$	*Not everybody is green.*
$\exists x\ \neg P(x)$	*There exists someone who is not green.*

Note that both statements are equivalent!

In general we have the following equivalences:

$$\neg\forall x\ \phi \iff \exists x\ \neg\phi$$

$$\neg\exists x\ \phi \iff \forall x\ \neg\phi$$

$$\forall x\ \phi \iff \neg\exists x\ \neg\phi$$

$$\exists x\ \phi \iff \neg\forall x\ \neg\phi$$

Note that all these equivalences follow from each other.

Predicate Logic: Syntax

The **atomic formulas** such as $P(a)$, $R(a, b)$,...,$P(x)$,$R(x, y)$ are the building blocks of formulas:

- here P, R, ... are the **predicate symbols**,
- $a, b, c, \ldots$ are **constants**,
- $x, y, z, \ldots$ are **variables**.

Complex formulas can be build from:

- atomic formulas,
- connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ to connect fomulas

$$\neg\phi \qquad \phi \vee \psi \qquad \phi \wedge \psi \qquad \phi \rightarrow \psi$$

- quantifiers $\forall x$, $\forall y$, ... and $\exists x$, $\exists y$, ...

$$\forall x\ \phi \qquad \exists x\ \phi$$

Brackets

Priority and Brackets

The priority rules for $\forall$ and $\exists$ are the same as for $\neg$.

$$(\forall x\ D(x)) \wedge R(a, j)$$

can be written as

$$\forall x\ D(x) \wedge R(a, j)$$

$$\forall x\ (D(x) \to R(a, x))$$

needs brackets! It is **not equivalent to**

$$\forall x\ D(x) \to R(a, x)$$

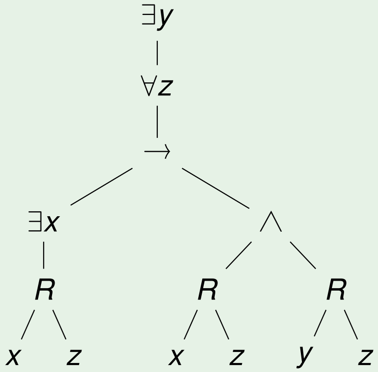
which is short for

$$(\forall x\ D(x)) \to R(a, x)$$

Parse Trees

We make a parse tree from the formula

$$\exists y\, \forall z\, (\exists x\, R(x,z) \rightarrow (R(x,z) \wedge R(y,z)))$$



The x on the left is in the scope of $\exists x$; the x in the middle not.

Bound and Free Variables

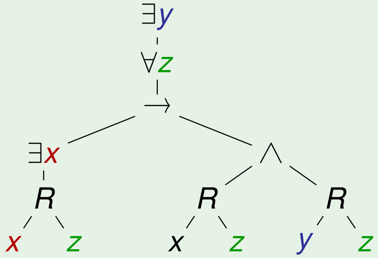
A quantifier **binds** the **free (not yet bound)** variables of the same name **in the scope** of the quantifier.

The $\forall x$ or $\exists x$ in

$$\forall x\ (\phi) \qquad\qquad \exists x\ (\phi)$$

binds all x in ϕ that are free, that is, not already bound.

$$\exists y\ \forall z\ (\exists x\ R(x, z) \rightarrow (R(x, z) \wedge R(y, z)))$$



The only black x is **not bound**, a **free variable**.

Exam Exercise

Indicate by arrows which variable is bound by which quantifier:

- $\exists x\ (\forall y\ R(x, y) \to R(y, x))$
- $\exists x\ \forall y\ (R(x, y) \to \exists x\ R(y, x) \to P(x))$

Substitution

Substitution of t for x in ϕ

$$\phi\ [t/x]$$

is the result of replacing all free occurrences of the **variable** x in ϕ with the **term** t.

- $\forall x\ P(x) \wedge Q(x)\ \ [y/x] \quad = \quad \forall x\ P(x) \wedge Q(y)$
- $\exists y\ R(x,y)\ \ [z/x] \quad = \quad \exists y\ R(z,y)$
- $\exists y\ R(x,y)\ \ [z/y] \quad = \quad \exists y\ R(x,y)$

Attention!

- $\exists y\ R(x,y)\ \ [y/x] \quad \neq \quad \exists y\ R(y,y)$

This substitution is **forbidden** since y is not **free for x** in $\exists y\ R(x,y)$.

Examples: Semantics Intuitive

Semantics Intuitive

The semantics of a predicate logic formula are **models.**

A **model** for a formula consists of

- a **universe** of objects/individuals
- **predicates** / **relations** over the universe

such that the formula holds in this model.

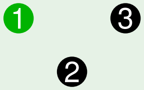
$$\exists x \; P(x)$$

This formula has a **model**; for example

- universe $\{1, 2, 3\}$
- $P(1),\ \neg P(2),\ \neg P(3)$

In a picture, this looks as follows:

- a **green dot** indicates that P is T



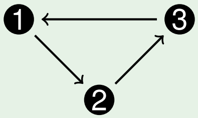
Examples: Semantics Intuitive

$$\forall x \, \exists y \, R(x, y)$$

This formula has a model; for example

- universe $\{1, 2, 3\}$
- $R(1, 2)$, $R(2, 3)$, $R(3, 1)$ is true ($R(x, y)$ is false otherwise)

We can draw this model as follows



Here an **arrow** from x to y means $R(x, y)$.

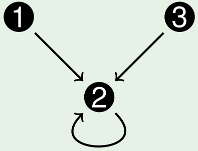
Is the following formula valid in this model?

$$\exists y \, \forall x \, R(x, y)$$

Examples: Semantics Intuitive

$$\exists y \, \forall x \; R(x, y)$$

Find a model for this formula!



Here the formula is true since for $y = 2$, we have: $\forall x \; R(x, y)$

- R(1,2)
- R(2,2)
- R(3,2)

Examples: Semantics Intuitive

$$\exists x\; P(x) \;\wedge\; \forall x\; \big(P(x) \to \exists y\; (Q(y) \wedge R(x,y))\big)$$

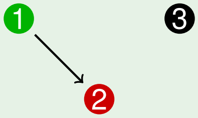
Assume the following meaning of the predicates:

- $P(x)$ = 'x is green',
- $Q(x)$ = 'x is red', and
- $R(x,y)$ = 'x knows y'.

What does the formula mean?

- there exists a green element, and
- every green element knows a red element.

Find a model for this formula!



Examples: Semantics Intuitive

Assume that

- $R(x, y)$ means 'x knows y'.

What is then the intuitive meaning of the following formulas?

- $\forall x \; \exists y \; R(x, y)$
  Everybody knows somebody.
- $\exists y \; \forall x \; R(x, y)$
  Somebody is known by everybody.
- $\forall y \; \exists x \; R(x, y)$
  Everybody is known by somebody.
- $\exists x \; \forall y \; R(x, y)$
  Somebody knows everybody.

For the exam you need to be able to interpret formulas!

Examples: Semantics Intuitive

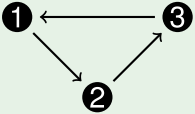
Typical exam question.

Which of the following semantic implications are true?

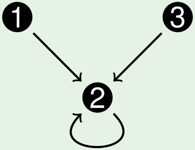
(a)	$\forall x\, \exists y\, R(x,y)$	$\models$	$\exists x\, \forall y\, R(x,y)$	*NO*
(b)	$\forall x\, \exists y\, R(x,y)$	$\models$	$\exists y\, \forall x\, R(x,y)$	*NO*
(c)	$\exists y\, \forall x\, R(x,y)$	$\models$	$\forall x\, \exists y\, R(x,y)$	*YES*
(d)	$\exists y\, \forall x\, R(x,y)$	$\models$	$\forall y\, \exists x\, R(x,y)$	*NO*

For the non-valid implications give counter-models.
(Models that make the premise true, the conclusion false.)

(a), (b)



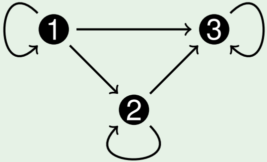
(d)



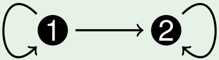
Thus only (c) is a valid semantic implication!

$$\forall x \, \forall y \, (R(x, y) \lor R(y, x))$$

Find a model for this formula!



Other examples of models for this formula:

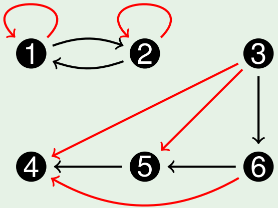


or a model with one element (universe is always non-empty):

Examples: Semantics Intuitive

$$\forall x \, \forall y \, \forall z \, (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

What arrows are missing to make the following a model?



(Add only those arrows that are really needed.)

Examples: Semantics Intuitive

This is a typical exam task!

$$\forall x\, \forall y\, \big(R(x,y) \vee R(y,x)\big)$$
$$\wedge \forall x\, \forall y\, \forall z\, \big(R(x,y) \wedge R(y,z) \to R(x,z)\big)$$
$$\wedge \forall x\, \exists y\, \neg R(x,y)$$

Find a model for this formula, or explain why there is none.

This formula has a model:

- universe $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \ldots\}$
- $R(x,y)$ if $x \geq y$

We check that the formula holds in the model:

- for all $x, y \in \mathbb{N}$, we have $x \geq y \vee y \geq x$
- for all $x, y, z \in \mathbb{N}$, we have $x \geq y \wedge y \geq z \to x \geq z$
- for every $x \in \mathbb{N}$ there is $y \in \mathbb{N}$ such that $x \not\geq y$