Logic and Modelling

— Introduction to Predicate Logic —

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- $L(x, y, z) = x, y, z$ are points on same line in the plane

Formulas and Logic Connectives

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Predicate logic is more expressive

In propositional logic, we could only state p, $p \wedge q$, $p \rightarrow \neg q$.

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Examples of formulas

- $\neg \forall x \; P(x)$
- $\exists x \; \neg P(x)$
- $\forall x \; \neg R(x, x)$
- $\forall x \; (R(x, x) \to \neg P(x))$
- $\forall x \; \forall y \; (R(x, y) \to \neg R(y, x))$

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  Not everybody is green.
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  Everybody, who knows himself, is not green.
- $\forall x \; \forall y \; (R(x, y) \to \neg R(y, x))$
  For all x and y, if x knows y then y does not know x.

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Not Forall and Not Exists

Let us reconsider two examples

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In general we have the following equivalences:

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$$\forall x\ \phi \iff \neg\exists x\ \neg\phi$$

$$\exists x\ \phi \iff \neg\forall x\ \neg\phi$$

Note that all these equivalences follow from each other.

Predicate Logic: Syntax

The **atomic formulas** such as $P(a)$, $R(a, b)$,...,$P(x)$,$R(x, y)$ are the building blocks of formulas:

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- quantifiers $\forall x$, $\forall y$, ... and $\exists x$, $\exists y$, ...

$$\forall x\ \phi \qquad \exists x\ \phi$$

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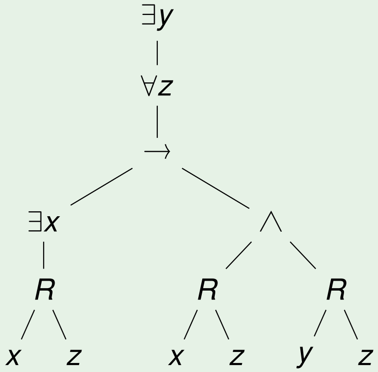

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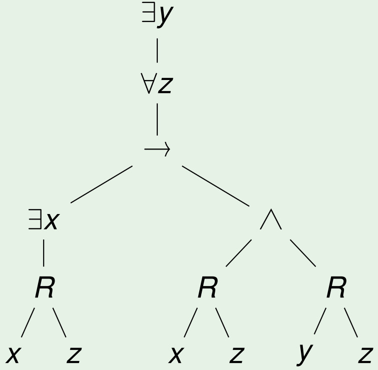

The x on the left is in the scope of $\exists x$; the x in the middle not.

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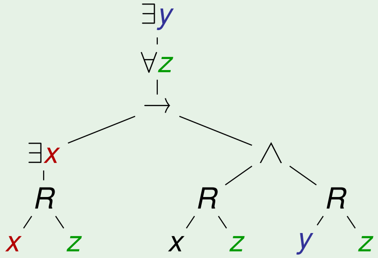

The only black x is **not bound**, a **free variable**.

Exam Exercise

Indicate by arrows which variable is bound by which quantifier:

- $\exists x \ (\forall y \ R(x, y) \rightarrow R(y, x))$
- $\exists x \ \forall y \ (R(x, y) \rightarrow \exists x \ R(y, x) \rightarrow P(x))$

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Attention!

- $\exists y\ R(x, y)\ \ [y/x] \quad \neq \quad \exists y\ R(y, y)$

This substitution is **forbidden** since y is not **free for x** in $\exists y\ R(x, y)$.

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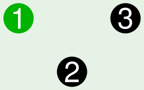
In a picture, this looks as follows:

- a **green dot** indicates that P is T



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- $R(1, 2)$, $R(2, 3)$, $R(3, 1)$ is true ($R(x, y)$ is false otherwise)

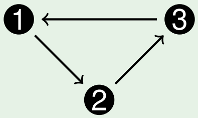
Examples: Semantics Intuitive

$$\forall x \, \exists y \; R(x, y)$$

This formula has a model; for example

- universe $\{1, 2, 3\}$
- $R(1, 2)$, $R(2, 3)$, $R(3, 1)$ is true ($R(x, y)$ is false otherwise)

We can draw this model as follows



Here an **arrow** from x to y means $R(x, y)$.

Examples: Semantics Intuitive

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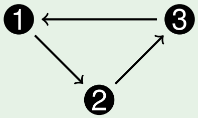

Here an **arrow** from x to y means $R(x, y)$.

Is the following formula valid in this model?

$$\exists y\ \forall x\ R(x, y)$$

Examples: Semantics Intuitive

$$\exists y \, \forall x \; R(x, y)$$

Examples: Semantics Intuitive

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Find a model for this formula!

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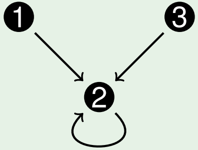
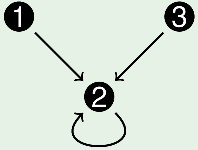


Examples: Semantics Intuitive

$$\exists y \; \forall x \; R(x, y)$$

Find a model for this formula!



Here the formula is true since for $y = 2$, we have: $\forall x \; R(x, y)$

- R(1,2)
- R(2,2)
- R(3,2)

Examples: Semantics Intuitive

$$\exists x\, P(x) \;\wedge\; \forall x\, \big(P(x) \rightarrow \exists y\, (Q(y) \wedge R(x, y))\big)$$

Examples: Semantics Intuitive

$$\exists x\ P(x)\ \wedge\ \forall x\ \big(P(x) \to \exists y\ (Q(y) \wedge R(x,y))\big)$$

Assume the following meaning of the predicates:

- $P(x)$ = 'x is green',
- $Q(x)$ = 'x is red', and
- $R(x,y)$ = 'x knows y'.

Examples: Semantics Intuitive

$$\exists x\ P(x)\ \wedge\ \forall x\ \big(P(x) \rightarrow \exists y\ (Q(y) \wedge R(x, y))\big)$$

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Examples: Semantics Intuitive

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- there exists a green element, and
- every green element knows a red element.

Examples: Semantics Intuitive

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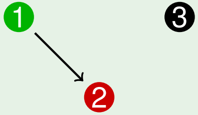

Examples: Semantics Intuitive

Assume that

- $R(x, y)$ means '*x* knows *y*'.

Examples: Semantics Intuitive

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What is then the intuitive meaning of the following formulas?

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- $\forall x\ \exists y\ R(x, y)$
  Everybody knows somebody.

Examples: Semantics Intuitive

Assume that

- $R(x, y)$ means 'x knows y'.

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  Everybody knows somebody.
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Examples: Semantics Intuitive

Assume that

- $R(x, y)$ means 'x knows y'.

What is then the intuitive meaning of the following formulas?

- $\forall x\ \exists y\ R(x, y)$
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- $\exists y\ \forall x\ R(x, y)$
  Somebody is known by everybody.

Examples: Semantics Intuitive

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  Somebody knows everybody.

For the exam you need to be able to interpret formulas!

Examples: Semantics Intuitive

Typical exam question.

Which of the following semantic implications are true?

(a) $\forall x \, \exists y \, R(x, y) \models \exists x \, \forall y \, R(x, y)$

(b) $\forall x \, \exists y \, R(x, y) \models \exists y \, \forall x \, R(x, y)$

(c) $\exists y \, \forall x \, R(x, y) \models \forall x \, \exists y \, R(x, y)$

(d) $\exists y \, \forall x \, R(x, y) \models \forall y \, \exists x \, R(x, y)$

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(d) $\exists y \, \forall x \; R(x,y) \models \forall y \, \exists x \; R(x,y)$

For the non-valid implications give counter-models.
(Models that make the premise true, the conclusion false.)

Examples: Semantics Intuitive

Typical exam question.

Which of the following semantic implications are true?

(a)	$\forall x\, \exists y\, R(x, y)$	$\models$	$\exists x\, \forall y\, R(x, y)$	*NO*
(b)	$\forall x\, \exists y\, R(x, y)$	$\models$	$\exists y\, \forall x\, R(x, y)$	
(c)	$\exists y\, \forall x\, R(x, y)$	$\models$	$\forall x\, \exists y\, R(x, y)$	
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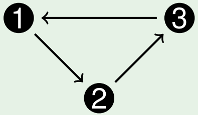

Examples: Semantics Intuitive

Typical exam question.

Which of the following semantic implications are true?

(a) $\forall x\, \exists y\, R(x, y) \models \exists x\, \forall y\, R(x, y)$ *NO*

(b) $\forall x\, \exists y\, R(x, y) \models \exists y\, \forall x\, R(x, y)$ *NO*

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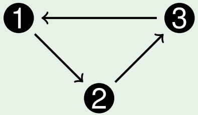

Examples: Semantics Intuitive

Typical exam question.

Which of the following semantic implications are true?

(a)	$\forall x \, \exists y \; R(x, y)$	$\models$	$\exists x \, \forall y \; R(x, y)$	*NO*
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(c)	$\exists y \, \forall x \; R(x, y)$	$\models$	$\forall x \, \exists y \; R(x, y)$	*YES*
(d)	$\exists y \, \forall x \; R(x, y)$	$\models$	$\forall y \, \exists x \; R(x, y)$	

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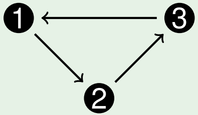

Examples: Semantics Intuitive

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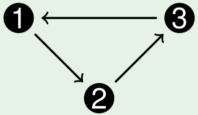
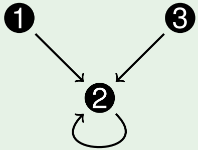


(d)



Examples: Semantics Intuitive

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(c)	$\exists y\, \forall x\, R(x,y)$	$\models$	$\forall x\, \exists y\, R(x,y)$	*YES*
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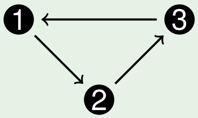
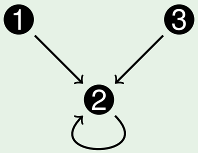


(d)



Thus only (c) is a valid semantic implication!

Examples: Semantics Intuitive

$$\forall x \, \forall y \, (R(x, y) \lor R(y, x))$$

Examples: Semantics Intuitive

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Find a model for this formula!

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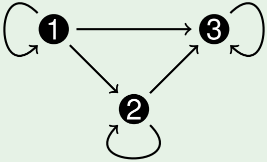

Examples: Semantics Intuitive

$$\forall x \, \forall y \, (R(x, y) \vee R(y, x))$$

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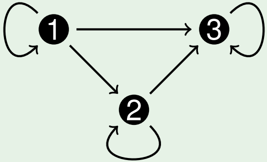
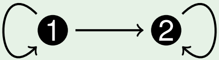


Other examples of models for this formula:



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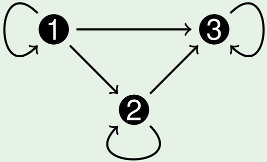

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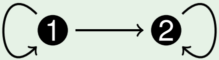

or a model with one element (universe is always non-empty):

Examples: Semantics Intuitive

$$\forall x \, \forall y \, \forall z \, (R(x, y) \land R(y, z) \rightarrow R(x, z))$$

Examples: Semantics Intuitive

$$\forall x\ \forall y\ \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$$

What arrows are missing to make the following a model?

$$\begin{tikzcd}
1 \arrow[r, bend left] & 2 \arrow[l, bend left] & 3 \arrow[d] \\
4 & 5 \arrow[l] & 6 \arrow[l]
\end{tikzcd}$$

(Add only those arrows that are really needed.)

Examples: Semantics Intuitive

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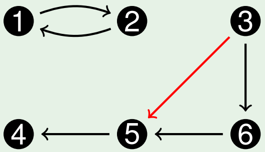

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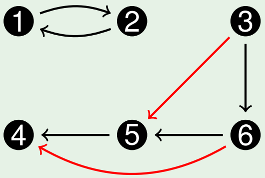

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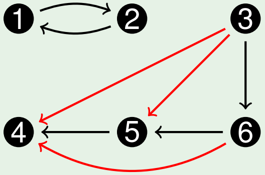

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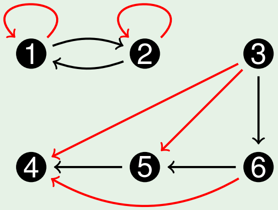

(Add only those arrows that are really needed.)

Examples: Semantics Intuitive

This is a typical exam task!

$$
\begin{aligned}
&\forall x\, \forall y\, \big(R(x,y) \lor R(y,x)\big) \\
&\land \forall x\, \forall y\, \forall z\, \big(R(x,y) \land R(y,z) \to R(x,z)\big) \\
&\land \forall x\, \exists y\, \neg R(x,y)
\end{aligned}
$$

Examples: Semantics Intuitive

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Find a model for this formula, or explain why there is none.

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Find a model for this formula, or explain why there is none.

This formula has a model:

- universe $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \ldots\}$
- $R(x, y)$ if $x \geq y$

Examples: Semantics Intuitive

This is a typical exam task!

$$\forall x\, \forall y\, \big(R(x,y) \vee R(y,x)\big)$$
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We check that the formula holds in the model:

Examples: Semantics Intuitive

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This formula has a model:

- universe $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \dots\}$
- $R(x,y)$ if $x \geq y$

We check that the formula holds in the model:

- for all $x, y \in \mathbb{N}$, we have $x \geq y \lor y \geq x$

Examples: Semantics Intuitive

This is a typical exam task!

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&\forall x\, \forall y\, \big(R(x,y) \vee R(y,x)\big) \\
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We check that the formula holds in the model:

- for all $x, y \in \mathbb{N}$, we have $x \geq y \vee y \geq x$
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This formula has a model:

- universe $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \ldots\}$
- $R(x,y)$ if $x \geq y$

We check that the formula holds in the model:

- for all $x, y \in \mathbb{N}$, we have $x \geq y \vee y \geq x$
- for all $x, y, z \in \mathbb{N}$, we have $x \geq y \wedge y \geq z \rightarrow x \geq z$
- for every $x \in \mathbb{N}$ there is $y \in \mathbb{N}$ such that $x \not\geq y$