

Logic and Modelling

— Introduction to Predicate Logic —

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- ▶ $L(x, y, z) = x, y, z$ are points on same line in the plane

Formulas and Logic Connectives

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Predicate logic is more expressive

In propositional logic, we could only state $p, p \wedge q, p \rightarrow \neg q$.

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- ▶ $\exists x \neg P(x)$
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Everybody, who knows himself, is not green.

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For all x and y , if x knows y then y does not know x .

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Not Forall and Not Exists

Let us reconsider two examples

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In general we have the following equivalences:

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$$\forall x \phi \iff \neg \exists x \neg \phi$$

$$\exists x \phi \iff \neg \forall x \neg \phi$$

Note that all these equivalences follow from each other.

Predicate Logic: Syntax

The **atomic formulas** such as $P(a)$, $R(a, b)$, \dots , $P(x)$, $R(x, y)$ are the building blocks of formulas:

- ▶ here P , R , \dots are the **predicate symbols**,
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- ▶ quantifiers $\forall x$, $\forall y$, \dots and $\exists x$, $\exists y$, \dots

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Priority and Brackets

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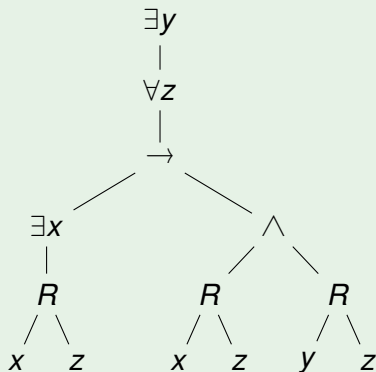
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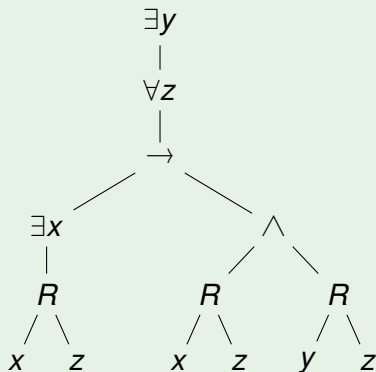
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The x on the left is in the scope of $\exists x$; the x in the middle not.

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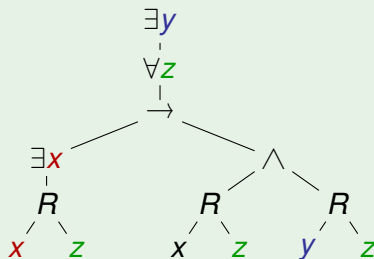
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The only black x is **not bound**, a **free variable**.

Exam Exercise

Indicate by arrows which variable is bound by which quantifier:

▶ $\exists x (\forall y R(x, y) \rightarrow R(y, x))$

▶ $\exists x \forall y (R(x, y) \rightarrow \exists x R(y, x) \rightarrow P(x))$

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Substitution of t for x in ϕ

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Attention!

- ▶ $\exists y R(x, y) [y/x] \neq \exists y R(y, y)$

This substitution is **forbidden** since y is not **free for** x in $\exists y R(x, y)$.

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In a picture, this looks as follows:

- ▶ a **green dot** indicates that P is T



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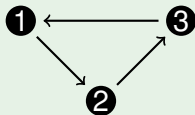
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We can draw this model as follows



Here an **arrow** from x to y means $R(x, y)$.

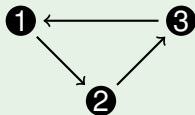
Examples: Semantics Intuitive

$$\forall x \exists y R(x, y)$$

This formula has a model; for example

- ▶ universe $\{1, 2, 3\}$
- ▶ $R(1, 2), R(2, 3), R(3, 1)$ is true ($R(x, y)$ is false otherwise)

We can draw this model as follows



Here an **arrow** from x to y means $R(x, y)$.

Is the following formula valid in this model?

$$\exists y \forall x R(x, y)$$

Examples: Semantics Intuitive

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Examples: Semantics Intuitive

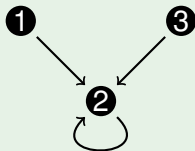
$$\exists y \forall x R(x, y)$$

Find a model for this formula!

Examples: Semantics Intuitive

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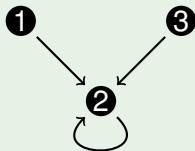
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Examples: Semantics Intuitive

$$\exists y \forall x R(x, y)$$

Find a model for this formula!



Here the formula is true since for $y = 2$, we have: $\forall x R(x, y)$

- ▶ $R(1,2)$
- ▶ $R(2,2)$
- ▶ $R(3,2)$

Examples: Semantics Intuitive

$$\exists x P(x) \wedge \forall x (P(x) \rightarrow \exists y (Q(y) \wedge R(x, y)))$$

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$$\exists x P(x) \wedge \forall x (P(x) \rightarrow \exists y (Q(y) \wedge R(x, y)))$$

Assume the following meaning of the predicates:

- ▶ $P(x)$ = 'x is green',
- ▶ $Q(x)$ = 'x is red', and
- ▶ $R(x, y)$ = 'x knows y'.

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- ▶ there exists a green element, and
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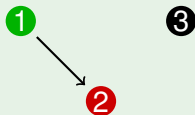
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Examples: Semantics Intuitive

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- ▶ $\forall x \exists y R(x, y)$

Everybody knows somebody.

Examples: Semantics Intuitive

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What is then the intuitive meaning of the following formulas?

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Somebody knows everybody.

For the exam you need to be able to interpret formulas!

Examples: Semantics Intuitive

Typical exam question.

Which of the following semantic implications are true?

$$(a) \quad \forall x \exists y R(x, y) \models \exists x \forall y R(x, y)$$

$$(b) \quad \forall x \exists y R(x, y) \models \exists y \forall x R(x, y)$$

$$(c) \quad \exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$$

$$(d) \quad \exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$$

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$$(d) \quad \exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$$

For the non-valid implications give counter-models.

(Models that make the premise true, the conclusion false.)

Examples: Semantics Intuitive

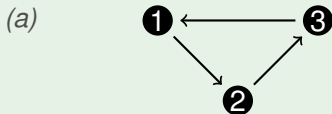
Typical exam question.

Which of the following semantic implications are true?

- (a) $\forall x \exists y R(x, y) \models \exists x \forall y R(x, y)$ **NO**
- (b) $\forall x \exists y R(x, y) \models \exists y \forall x R(x, y)$
- (c) $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$
- (d) $\exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$

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Examples: Semantics Intuitive

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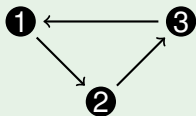
(c) $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$

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(a), (b)



Examples: Semantics Intuitive

Typical exam question.

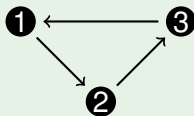
Which of the following semantic implications are true?

- (a) $\forall x \exists y R(x, y) \models \exists x \forall y R(x, y)$ **NO**
- (b) $\forall x \exists y R(x, y) \models \exists y \forall x R(x, y)$ **NO**
- (c) $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$ **YES**
- (d) $\exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$

For the non-valid implications give counter-models.

(Models that make the premise true, the conclusion false.)

(a), (b)



Examples: Semantics Intuitive

Typical exam question.

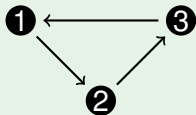
Which of the following semantic implications are true?

- (a) $\forall x \exists y R(x, y) \models \exists x \forall y R(x, y)$ **NO**
- (b) $\forall x \exists y R(x, y) \models \exists y \forall x R(x, y)$ **NO**
- (c) $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$ **YES**
- (d) $\exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$ **NO**

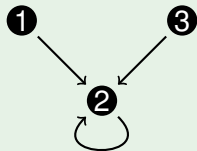
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(d)



Examples: Semantics Intuitive

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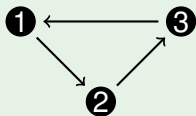
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- (d) $\exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$ **NO**

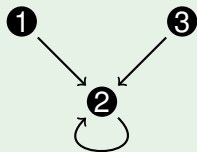
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(d)



Thus only (c) is a valid semantic implication!

Examples: Semantics Intuitive

$$\forall x \forall y (R(x, y) \vee R(y, x))$$

Examples: Semantics Intuitive

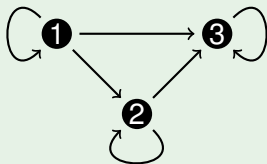
$$\forall x \forall y (R(x, y) \vee R(y, x))$$

Find a model for this formula!

Examples: Semantics Intuitive

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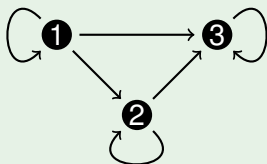
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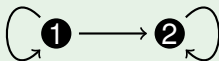
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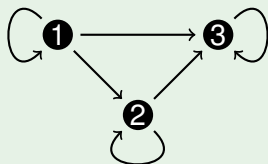
Other examples of models for this formula:



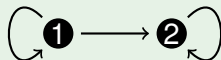
Examples: Semantics Intuitive

$$\forall x \forall y (R(x, y) \vee R(y, x))$$

Find a model for this formula!



Other examples of models for this formula:



or a model with one element (universe is always non-empty):



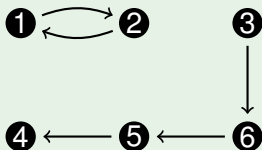
Examples: Semantics Intuitive

$$\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

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What arrows are missing to make the following a model?

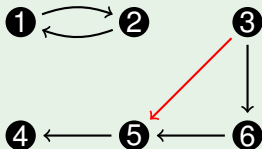


(Add only those arrows that are really needed.)

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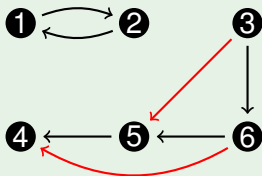


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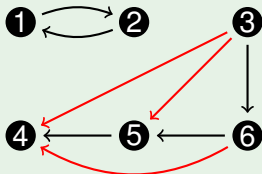


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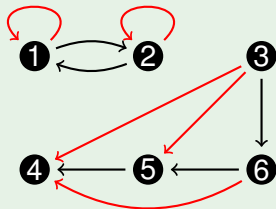


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Examples: Semantics Intuitive

This is a typical exam task!

$$\forall x \forall y (R(x, y) \vee R(y, x))$$

$$\wedge \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

$$\wedge \forall x \exists y \neg R(x, y)$$

Examples: Semantics Intuitive

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$$\wedge \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

$$\wedge \forall x \exists y \neg R(x, y)$$

Find a model for this formula, or explain why there is none.

Examples: Semantics Intuitive

This is a typical exam task!

$$\begin{aligned} & \forall x \forall y (R(x, y) \vee R(y, x)) \\ & \wedge \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \\ & \wedge \forall x \exists y \neg R(x, y) \end{aligned}$$

Find a model for this formula, or explain why there is none.

This formula has a model:

- ▶ universe $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \dots\}$
- ▶ $R(x, y)$ if $x \geq y$

Examples: Semantics Intuitive

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We check that the formula holds in the model:

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- ▶ for every $x \in \mathbb{N}$ there is $y \in \mathbb{N}$ such that $x \not\geq y$