

# Logic and Modelling

— Propositional Logic: Correctness and Completeness —

Jörg Endrullis

VU University Amsterdam

# Truth Values

In propositional logic, the **truth value** of a formula is determined by a **truth assignment** of the variables in the formula.

For example, assigning

- ▶ to  $p$  and  $q$  the truth value T, and
- ▶ to  $r$  the truth value F,

determines the truth value of the formula  $p \vee \neg q \rightarrow r$ .

The truth value of a formula can, for example, be computed via:

- ▶ the **parse tree**, or
- ▶ by making a **truth table**.

# Valuations

An **assignment of truth values** to variables is a **valuation**.

(In the book, for predicate logic, also called look-up function)

The valuation  $\ell$  on the previous slide is

$$\ell(p) = T \qquad \ell(q) = T \qquad \ell(r) = F$$

The truth value of  $p \vee \neg q \rightarrow r$  with this valuation is F.

- ▶ A valuation corresponds to one line in the truth table.
- ▶ A truth table systemically considers all possible valuations.

For  $p \vee \neg q \rightarrow r$ , there are  $8 = 2^3$  valuations.

# Semantic and Syntactic Reasoning

## Semantic implication (or semantic entailment)

$$\phi_1, \dots, \phi_n \models \psi$$

means

Every valuation that makes  $\phi_1, \dots, \phi_n$  true,  
also makes  $\psi$  true.

(Recall, valuation is interpretation of the propositional letters.)

## Syntactic derivability

$$\phi_1, \dots, \phi_n \vdash \psi$$

means

There is a natural deduction derivation of  $\psi$   
starting from premises  $\phi_1, \dots, \phi_n$ .

# The relation of $\models$ and $\vdash$

## Theorem

The **semantic entailment**  $\models$  and **syntactic derivability**  $\vdash$  coincide:

$$\phi_1, \dots, \phi_n \models \psi \iff \phi_1, \dots, \phi_n \vdash \psi$$

## Two directions

$\Rightarrow$  **Completeness**

In a slogan: everything true is derivable.

$\Leftarrow$  **Soundness (or Correctness)**

In a slogan: everything derivable is true.

# Correctness Theorem

## Soundness / Correctness

$$\phi_1, \dots, \phi_n \vdash \psi \quad \Rightarrow \quad \phi_1, \dots, \phi_n \models \psi$$

## Explanation

In a slogan: everything derivable is true.

If  $\psi$  is syntactically derivable from  $\phi_1, \dots, \phi_n$ ,  
then every valuation that makes  $\phi_1, \dots, \phi_n$  true, makes  $\psi$  true.

Thus truth in a model (valuation) is preserved under derivation.

The syntactic deduction rules are **correct** in the sense that it is not possible to derive **false conclusions** from **true premises**.

# Completeness Theorem

## Completeness

$$\phi_1, \dots, \phi_n \models \psi \quad \Rightarrow \quad \phi_1, \dots, \phi_n \vdash \psi$$

## Explanation

In a slogan: everything true is derivable.

If  $\psi$  follows semantically from  $\phi_1, \dots, \phi_n$ ,  
then  $\psi$  can be derived syntactically from premises  $\phi_1, \dots, \phi_n$ .

This means that the syntactic derivation rules are strong enough to derive every semantic conclusion.

Thus the system is **complete**: no more rules are needed.

# Completeness and Correctness

**Completeness** and **correctness** are fundamental concepts.

They are important whenever a semantical notion is modelled using a formal rule system.