Logic and Modelling

— Propositional Logic: Correctness and Completeness —

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Truth Values

In propositional logic, the **truth value** of a formula is determined by a **truth assignment** of the variables in the formula.

For example, assigning

- to p and q the truth value T, and
- to r the truth value F,

determines the truth value of the formula $p \vee \neg q \to r$.

The truth value of a formula can, for example, be computed via:

- the **parse tree**, or
- by making a **truth table**.

Valuations

An **assignment of truth values** to variables is a **valuation**.

(In the book, for predicate logic, also called look-up function)

The valuation ℓ on the previous slide is

$$\ell(p) = \mathsf{T} \qquad \ell(q) = \mathsf{T} \qquad \ell(r) = \mathsf{F}$$

The truth value of $p \vee \neg q \to r$ with this valuation is F.

- A valuation corresponds to one line in the truth table.
- A truth table systemically considers all possible valuations.

For $\quad p \vee \neg q \to r, \quad$ there are $8 = 2^3$ valuations.

Semantic and Syntactic Reasoning

Semantic implication (or semantic entailment)

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Every valuation that makes $\phi_1, \ldots, \phi_n$ true,
also makes ψ true.

(Recall, valuation is interpretation of the propositional letters.)

Syntactic derivability

$$\phi_1, \ldots, \phi_n \vdash \psi$$

means

There is a natural deduction derivation of ψ
starting from premises $\phi_1, \ldots, \phi_n$.

The relation of $\models$ and $\vdash$

Theorem

The **semantic entailment** $\models$ and **syntactic derivability** $\vdash$ coincide:

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_n \vdash \psi$$

Two directions

$\Rightarrow$ **Completeness**
In a slogan: everything true is derivable.

$\Leftarrow$ **Soundness (or Correctness)**
In a slogan: everything derivable is true.

Correctness Theorem

Soundness / Correctness

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \Rightarrow \quad \phi_1, \ldots, \phi_n \models \psi$$

Explanation

In a slogan: everything derivable is true.

If ψ is syntactically derivable from $\phi_1, \ldots, \phi_n$,
then every valuation that makes $\phi_1, \ldots, \phi_n$ true, makes ψ true.

Thus truth in a model (valuation) is preserved under derivation.

The syntactic deduction rules are **correct** in the sense that it is not possible to derive **false conclusions** from **true premises**.

Completeness Theorem

Completeness

$$\phi_1, \ldots, \phi_n \models \psi \quad \Rightarrow \quad \phi_1, \ldots, \phi_n \vdash \psi$$

Explanation

In a slogan: everything true is derivable.

If ψ follows semantically from $\phi_1, \ldots, \phi_n$,
then ψ can be derived syntactically from premises $\phi_1, \ldots, \phi_n$.

This means that the syntactic derivation rules are strong enough to derive every semantic conclusion.

Thus the system is **complete**: no more rules are needed.

Completeness and Correctness

Completeness and **correctness** are fundamental concepts.

They are important whenever a semantical notion is modelled using a formal rule system.