Logic and Modelling

— Propositional Logic: Correctness and Completeness —

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determines the truth value of the formula $p \vee \neg q \to r$.

The truth value of a formula can, for example, be computed via:

- the **parse tree**, or
- by making a **truth table**.

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For $\quad p \vee \neg q \to r, \quad$ there are $8 = 2^3$ valuations.

Semantic and Syntactic Reasoning

Semantic implication (or semantic entailment)

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Every valuation that makes $\phi_1, \ldots, \phi_n$ true,
also makes ψ true.

(Recall, valuation is interpretation of the propositional letters.)

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Syntactic derivability

$$\phi_1, \ldots, \phi_n \vdash \psi$$

means

There is a natural deduction derivation of ψ
starting from premises $\phi_1, \ldots, \phi_n$.

The relation of $\models$ and $\vdash$

Theorem

The **semantic entailment** $\models$ and **syntactic derivability** $\vdash$ coincide:

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The syntactic deduction rules are **correct** in the sense that it is not possible to derive **false conclusions** from **true premises**.

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This means that the syntactic derivation rules are strong enough to derive every semantic conclusion.

Thus the system is **complete**: no more rules are needed.

Completeness and Correctness

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They are important whenever a semantical notion is modelled using a formal rule system.