

Logic and Modelling

— Natural Deduction for Propositional Logic —

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Natural Deduction

$$\phi_1, \dots, \phi_n \vdash \psi$$

means: there exists a **natural deduction derivation** with

- ▶ premises ϕ_1, \dots, ϕ_n , and
- ▶ conclusion ψ .

Natural deduction is a **formal system** with **strict formal rules!**

Rules for \wedge and \vee

Introduction of \wedge

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge_i$$

(If you have derived ϕ and ψ , then you can conclude $\phi \wedge \psi$.)

Elimination of \wedge

$$\frac{\phi \wedge \psi}{\phi} \wedge_{e_1}$$

$$\frac{\phi \wedge \psi}{\psi} \wedge_{e_2}$$

Rules for \vee

$$\frac{\phi}{\phi \vee \psi} \vee_{i_1}$$

$$\frac{\psi}{\phi \vee \psi} \vee_{i_2}$$

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1	$(p \wedge q) \wedge \neg r$	premise
2	$p \wedge q$	\wedge_{e_1} 1
3	q	\wedge_{e_2} 2
4	$\neg r$	\wedge_{e_2} 1
5	$q \wedge \neg r$	\wedge_i 3, 4

Hence we have derived

$$(p \wedge q) \wedge \neg r \vdash q \wedge \neg r$$

Rules for $\neg\neg$ and \rightarrow

Rules for $\neg\neg$

$$\frac{\neg\neg\phi}{\phi} \quad \neg\neg_e$$

$$\frac{\phi}{\neg\neg\phi} \quad \neg\neg_i$$

Elimination rules for \rightarrow

This rule is called “Modus Ponens” (MP):

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \quad \rightarrow_e \text{ (or } MP)$$

This rule is called “Modus Tollens” (MT):

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} \quad MT$$

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r$?

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \rightarrow r$	\rightarrow_e 4,1
6	$\neg\neg q$	MT 5,3
7	q	$\neg\neg_e$ 6

Hence we have derived

$$\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$$

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow_i$$

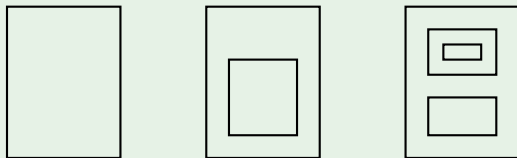
Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	\rightarrow_i 2-3

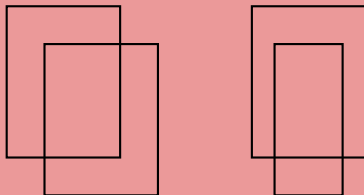
Block Structures

Allowed block structures

Blocks are allowed to be nested inside each other:



Blocks are not allowed to intersect:



Block Structure

When applying a rule

$$\frac{\phi_1 \quad \dots \quad \phi_n}{\psi},$$

the ϕ_1, \dots, ϕ_n must be **in the scope**, that is, must have been derived in the current block or a surrounding block.

(Compare with scopes of variables in programming languages.)

1	$p \rightarrow q$	premise
2	p	assumption
3	q	\rightarrow_e 1,2
4	$q \vee q$	\vee_i 3 This is not allowed!!!

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	\rightarrow_e 2,1
5	$\neg q$	MT 4,3
6	$\neg r \rightarrow \neg q$	\rightarrow_i 3-5

Hence we have derived

$$p \rightarrow (q \rightarrow r), p \vdash \neg r \rightarrow \neg q$$

Special Cases

1	p	assumption
2	$p \rightarrow p$	\rightarrow_i 1-1

This is a derivation of

$$\vdash p \rightarrow p$$

Copy Rule

Copy rule

$$\frac{\phi}{\phi} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	\rightarrow_i 2-3
5	$p \rightarrow (q \rightarrow p)$	\rightarrow_i 1-4

This concludes the derivation.

Rules for \neg and \perp

Rules for \neg and \perp

$$\frac{\phi \quad \neg\phi}{\perp} \neg_e$$

$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \perp \end{array}}}{\neg\phi} \neg_i$$

$$\frac{\perp}{\phi} \perp_e$$

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption
9	r	\rightarrow_e 8,7
10	\perp	\neg_e 9,3
11	$\neg\neg q$	\neg_i 8–10
12	q	$\neg\neg_e$ 11

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\phi$ from ϕ .

We can derive it using other rules as follows:

1	ϕ	premise
2	$\neg\phi$	assumption
3	\perp	\neg_e 1,2
4	$\neg\neg\phi$	\neg_i 2–3

Thus: $\phi \vdash \neg\neg\phi$.

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\phi$ from $\phi \rightarrow \psi$ and $\neg\psi$.

We can derive it using other rules as follows:

1	$\phi \rightarrow \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	\rightarrow_e 3,1
5	\perp	\neg_e 2,4
6	$\neg\phi$	\neg_i 3–5

Thus: $\phi \rightarrow \psi, \neg\psi \vdash \neg\phi$.

This shows that $\neg\neg_i$ and MT are **not needed**, but sometimes help to make derivations easier or shorter.

Elimination of \vee

Elimination of \vee

$$\frac{\phi \vee \psi \quad \begin{array}{|c|} \hline \phi \\ \vdots \\ \xi \\ \hline \end{array} \quad \begin{array}{|c|} \hline \psi \\ \vdots \\ \xi \\ \hline \end{array}}{\xi} \vee_e$$

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5
7	p	\vee_e 1, 3-3, 4-6

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4
6	q	assumption
7	q	\vee_e 1, 3–5, 6–6
8	$p \rightarrow q$	\rightarrow_i 2–7

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\phi \\ \vdots \\ \perp \end{array}$$

Then also

$$\boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \perp \end{array}}$$
$$\begin{array}{cc} \neg\neg\phi & \neg_i \\ \phi & \neg\neg_e \end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Also known as **Reductio ad Absurdum (RAA)**!

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \perp \end{array}}}{\phi} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise
2	$\neg\phi$	assumption
3	\perp	\neg_e 2,1
4	ϕ	PBC 2-3

Law of Excluded Middle

Law of Excluded Middle Rule

$$\overline{\phi \vee \neg\phi} \quad \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4
6	$\neg p$	assumption
7	$\neg p \vee q$	\vee_{i_1} 7
8	$\neg p \vee q$	\vee_e 2, 3–5, 6–7

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	$\neg\phi$	\neg_i 2–4
6	$\phi \vee \neg\phi$	\vee_{i_2} 5
7	\perp	\neg_e 6,1
8	$\phi \vee \neg\phi$	PBC 1–7

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

LEM

2 q

assumption

3 p

assumption

4 q

copy 2

5 $p \rightarrow q$

\rightarrow_i 3–4

6 $\neg q \vee (p \rightarrow q)$

\vee_{i_2} 5

7 $\neg q$

assumption

8 $\neg q \vee (p \rightarrow q)$

\vee_{i_1} 6

9 $\neg q \vee (p \rightarrow q)$

\vee_e 1, 2–6, 7–8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	\rightarrow_i 6–7
9	$\neg q \vee (p \rightarrow q)$	\vee_i 8
10	\perp	\neg_e 9,1
11	$\neg q \vee (p \rightarrow q)$	PBC 1–11

More Exam Preparation Tasks

Exam Exercises

Try to derive yourself:

- ▶ $p \vee q, \neg p \vdash q$
- ▶ $p \rightarrow (q \rightarrow r) \vdash q \rightarrow (p \rightarrow r)$
- ▶ $(p \rightarrow q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$
- ▶ $p \vee (q \wedge r) \vdash p \vee q$
- ▶ $a \vee b, a \rightarrow c, \neg d \rightarrow \neg b \vdash c \vee d$
- ▶ $(a \rightarrow b) \wedge (b \rightarrow a) \vdash (a \wedge b) \vee (\neg a \wedge \neg b)$
- ▶ $a \wedge (b \vee c) \vdash (a \wedge b) \vee (a \wedge c)$
- ▶ $\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$