Logic and Modelling

— Natural Deduction for Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Natural Deduction

$$\phi_1, \ldots, \phi_n \vdash \psi$$

means: there exists a **natural deduction derivation** with

- premises $\phi_1, \ldots, \phi_n$, and
- conclusion ψ.

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- conclusion ψ.

Natural deduction is a **formal system** with **strict formal rules**!

Rules for $\wedge$ and $\vee$

Introduction of $\wedge$

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \quad \wedge_i$$

(If you have derived ϕ and ψ, then you can conclude $\phi \wedge \psi$.)

Rules for $\wedge$ and $\vee$

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Elimination of $\wedge$

$$\frac{\phi \wedge \psi}{\phi} \; \wedge_{e_1} \qquad\qquad \frac{\phi \wedge \psi}{\psi} \; \wedge_{e_2}$$

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Rules for $\vee$

$$\frac{\phi}{\phi \vee \psi} \; \vee_{i_1} \qquad\qquad \frac{\psi}{\phi \vee \psi} \; \vee_{i_2}$$

Derivation with Natural Deduction

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Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

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2	$p \wedge q$	$\wedge_{e_1}$ 1
3	q	$\wedge_{e_2}$ 2

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3	q	$\wedge_{e_2}$ 2
4	$\neg r$	$\wedge_{e_2}$ 1

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3	q	$\wedge_{e_2}$ 2
4	$\neg r$	$\wedge_{e_2}$ 1
5	$q \wedge \neg r$	$\wedge_i$ 3, 4

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2	$p \wedge q$	$\wedge_{e_1}$ 1
3	q	$\wedge_{e_2}$ 2
4	$\neg r$	$\wedge_{e_2}$ 1
5	$q \wedge \neg r$	$\wedge_i$ 3, 4

Hence we have derived

$$(p \wedge q) \wedge \neg r \quad \vdash \quad q \wedge \neg r$$

Rules for $\neg\neg$ and $\rightarrow$

Rules for $\neg\neg$

$$\frac{\neg\neg\phi}{\phi} \quad \neg\neg_e \qquad\qquad \frac{\phi}{\neg\neg\phi} \quad \neg\neg_i$$

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$$\frac{\neg\neg\phi}{\phi}\ \neg\neg_e \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg_i$$

Elimination rules for $\rightarrow$

This rule is called “Modus Ponens” (MP):

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}\ \rightarrow_e\ (\mathit{or}\ \ \mathit{MP})$$

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This rule is called "Modus Tollens" (MT):

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \textit{MT}$$

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1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \to (\neg q \to r),\ p,\ \neg r$?

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2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2

Derivation with Natural Deduction

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1	$\neg\neg p \to (\neg q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \to r$	$\to_e$ 4,1

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1	$\neg\neg p \to (\neg q \to r)$	premise
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4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \to r$	$\to_e$ 4,1
6	$\neg\neg q$	MT 5,3

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2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \to r$	$\to_e$ 4,1
6	$\neg\neg q$	MT 5,3
7	q	$\neg\neg_e$ 6

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Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r$?

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
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4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \rightarrow r$	$\rightarrow_e$ 4,1
6	$\neg\neg q$	MT 5,3
7	q	$\neg\neg_e$ 6

Hence we have derived

$$\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r \quad \vdash \quad q$$

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Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

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Derivation of $\quad p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption

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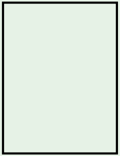
4	$\neg q \rightarrow \neg p$	$\rightarrow_i$ 2–3

Block Structures

Allowed block structures

Blocks are allowed to be nested inside each other:



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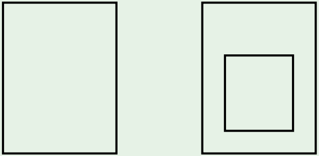

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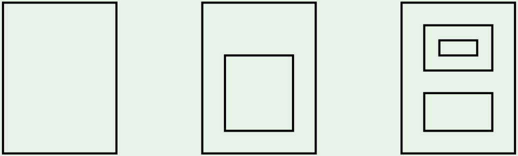

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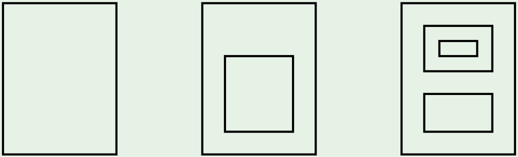
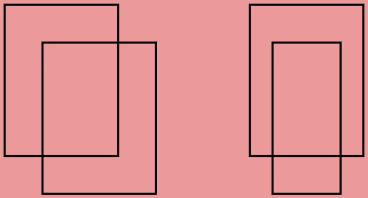


Blocks are not allowed to intersect:



Block Structure

When applying a rule

$$\frac{\phi_1 \qquad \dots \qquad \phi_n}{\psi} \quad ,$$

the $\phi_1, \dots, \phi_n$ must be **in the scope**, that is, must have been derived in the current block or a surrounding block.

(Compare with scopes of variables in programming languages.)

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(Compare with scopes of variables in programming languages.)

1	$p \rightarrow q$	premise
2	p	assumption
3	q	$\rightarrow_e$ 1,2
4	$q \vee q$	$\vee_i$ 3 **This is not allowed!!!**

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Derivation with Natural Deduction

Can we derive $\neg r \to \neg q$ from $p \to (q \to r),\ p$?

1	$p \to (q \to r)$	premise
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3	$\neg r$	assumption

Derivation with Natural Deduction

Can we derive $\neg r \to \neg q$ from $p \to (q \to r),\ p$?

1	$p \to (q \to r)$	premise
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3	$\neg r$	assumption
4	$q \to r$	$\to_e$ 2,1

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4	$q \rightarrow r$	$\rightarrow_e$ 2,1
5	$\neg q$	MT 4,3

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4	$q \to r$	$\to_e$ 2,1
5	$\neg q$	MT 4,3
6	$\neg r \to \neg q$	$\to_i$ 3–5

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Can we derive $\neg r \to \neg q$ from $p \to (q \to r),\ p$?

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3		$\neg r$	assumption
4		$q \to r$	$\to_e$ 2,1
5		$\neg q$	MT 4,3
6		$\neg r \to \neg q$	$\to_i$ 3–5

Hence we have derived

$$p \to (q \to r),\ p \quad \vdash \quad \neg r \to \neg q$$

Special Cases

1	$\boxed{p}$	assumption
2	$p \to p$	$\to_i$ 1–1

This is a derivation of

$$\vdash \; p \to p$$

Copy Rule

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$$\frac{\phi}{\phi} \quad \textit{copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

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Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	$\boxed{p}$	assumption

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1	p	assumption

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Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption

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Lets try to prove $\quad \vdash p \rightarrow (q \rightarrow p) \quad$!

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2	q	assumption
3	p	copy 1

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1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	$\rightarrow_i$ 2–3

Copy Rule

Copy rule

$$\frac{\phi}{\phi}\ \textit{copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	$\rightarrow_i$ 2–3
5	$p \rightarrow (q \rightarrow p)$	$\rightarrow_i$ 1–4

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This concludes the derivation.

Rules for ¬ and ⊥

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$$\frac{\phi \qquad \neg\phi}{\bot}\ \neg_e$$

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$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi} \; \neg i$$

Rules for ¬ and ⊥

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$$\frac{\phi \qquad \neg\phi}{\bot}\ \neg e$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg i$$

$$\frac{\bot}{\phi}\ \bot e$$

Example

Prove $\neg\neg p \to (\neg q \to r),\ p,\ \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

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2	p	premise

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1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	$\perp$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

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2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	$\perp$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \rightarrow r$	$\rightarrow_e$ 6,1

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4	$\neg p$	assumption
5	$\bot$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \to r$	$\to_e$ 6,1
8	$\neg q$	assumption

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Prove $\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	$\bot$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \rightarrow r$	$\rightarrow_e$ 6,1
8	$\neg q$	assumption
9	r	$\rightarrow_e$ 8,7

Example

Prove $\neg\neg p \to (\neg q \to r),\ p,\ \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \to (\neg q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\boxed{\neg p}$	assumption
5	$\boxed{\bot}$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \to r$	$\to_e$ 6,1
8	$\boxed{\neg q}$	assumption
9	$\boxed{r}$	$\to_e$ 8,7
10	$\boxed{\bot}$	$\neg_e$ 9,3

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5	$\bot$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \to r$	$\to_e$ 6,1
8	$\neg q$	assumption
9	r	$\to_e$ 8,7
10	$\bot$	$\neg_e$ 9,3
11	$\neg\neg q$	$\neg_i$ 8–10

Example

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5	$\bot$	$\neg_e$ 2,4
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11	$\neg\neg q$	$\neg_i$ 8–10
12	q	$\neg\neg_e$ 11

MT and $\neg\neg_i$ as "derived rules"

The $\neg\neg_i$ rule derives $\neg\neg\phi$ from ϕ.

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2	$\neg\phi$	assumption
3	$\bot$	$\neg_e$ 1,2

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We can derive it using other rules as follows:

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\perp$	$\neg_e$ 1,2
4	$\neg\neg\phi$	$\neg_i$ 2–3

Thus: $\phi \vdash \neg\neg\phi$.

MT and $\neg\neg_i$ as "derived rules"

The Modus Tollens rule derives $\neg\phi$ from $\phi \rightarrow \psi$ and $\neg\psi$.

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2	$\neg\psi$	premise

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We can derive it using other rules as follows:

1	$\phi \rightarrow \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption

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We can derive it using other rules as follows:

1	$\phi \to \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\to_e$ 3,1

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The Modus Tollens rule derives $\neg\phi$ from $\phi \rightarrow \psi$ and $\neg\psi$.

We can derive it using other rules as follows:

1	$\phi \rightarrow \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\rightarrow_e$ 3,1
5	$\bot$	$\neg_e$ 2,4

MT and $\neg\neg_i$ as "derived rules"

The Modus Tollens rule derives $\neg\phi$ from $\phi \rightarrow \psi$ and $\neg\psi$.

We can derive it using other rules as follows:

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2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\rightarrow_e$ 3,1
5	$\perp$	$\neg_e$ 2,4
6	$\neg\phi$	$\neg_i$ 3–5

MT and $\neg\neg_i$ as "derived rules"

The Modus Tollens rule derives $\neg\phi$ from $\phi \to \psi$ and $\neg\psi$.

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1	$\phi \to \psi$	premise
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3	ϕ	assumption
4	ψ	$\to_e$ 3,1
5	$\bot$	$\neg_e$ 2,4
6	$\neg\phi$	$\neg_i$ 3–5

Thus: $\phi \to \psi,\ \neg\psi \vdash \neg\phi$.

MT and $\neg\neg_i$ as "derived rules"

The Modus Tollens rule derives $\neg\phi$ from $\phi \to \psi$ and $\neg\psi$.

We can derive it using other rules as follows:

1	$\phi \to \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\to_e$ 3,1
5	$\perp$	$\neg_e$ 2,4
6	$\neg\phi$	$\neg_i$ 3–5

Thus: $\phi \to \psi, \neg\psi \vdash \neg\phi$.

This shows that $\neg\neg_i$ and MT are **not needed**, but sometimes help to make derivations easier or shorter.

Elimination of $\vee$

Elimination of $\vee$

$$\dfrac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots \\ \xi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots \\ \xi\end{array}}}{\xi} \; \vee_e$$

Example

Prove $\quad p \vee \neg q,\ \neg p \rightarrow q \ \vdash\ p \quad$!

Example

Prove $p \vee \neg q,\ \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise

Example

Prove $p \vee \neg q, \neg p \to q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \to q$	premise

Example

Prove $\quad p \vee \neg q,\ \neg p \rightarrow q \ \vdash\ p \quad$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise

Example

Prove $p \vee \neg q,\ \neg p \rightarrow q \ \vdash\ p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	$\boxed{p}$	assumption

Example

Prove $p \vee \neg q,\ \neg p \to q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \to q$	premise
3	p	assumption

Example

Prove $p \vee \neg q,\ \neg p \rightarrow q \ \vdash\ p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption

Example

Prove $p \vee \neg q,\ \neg p \to q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \to q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4

Example

Prove $p \vee \neg q,\ \neg p \to q \ \vdash\ p$!

1	$p \vee \neg q$	premise
2	$\neg p \to q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5

Example

Prove $p \vee \neg q,\ \neg p \to q \ \vdash\ p$!

1	$p \vee \neg q$	premise
2	$\neg p \to q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5
7	p	$\vee_e$ 1, 3–3, 4–6

Example

Use $\bot_e$ to prove $\quad \neg p \vee q \vdash p \rightarrow q \quad$!

Example

Use $\perp_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise

Example

Use $\perp_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption

Example

Use $\perp_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption

Example

Use $\perp_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\perp$	$\neg_e$ 2,3

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	$\neg_e$ 2,3
5	q	$\bot_e$ 4

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	$\neg_e$ 2,3
5	q	$\bot_e$ 4

Example

Use $\perp_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\perp$	$\neg_e$ 2,3
5	q	$\perp_e$ 4
6	q	assumption

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	$\neg_e$ 2,3
5	q	$\bot_e$ 4
6	q	assumption
7	q	$\vee_e$ 1, 3–5, 6–6

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	$\neg_e$ 2,3
5	q	$\bot_e$ 4
6	q	assumption
7	q	$\vee_e$ 1, 3–5, 6–6
8	$p \rightarrow q$	$\rightarrow_i$ 2–7

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}$$

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}$$

Then also

$$\begin{array}{ll} \boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}} & \\ \neg\neg\phi & \neg_i \\ \phi & \neg\neg_e \end{array}$$

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}$$

Then also

$$\begin{array}{ll} \boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}} & \\ \neg\neg\phi & \neg_i \\ \phi & \neg\neg_e \end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}$$

Then also

$$\begin{array}{ll} \boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}} & \\ \neg\neg\phi & \neg_i \\ \phi & \neg\neg_e \end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Also known as **Reductio ad Absurdum (RAA)**!

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\dfrac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \textit{PBC} \quad (\textit{or}\ \textit{RAA})$$

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \textit{PBC} \quad (\textit{or}\ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \mathit{PBC} \quad (\mathit{or}\ \ \mathit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \textit{PBC} \quad (\textit{or}\ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise

$$\boxed{}$$

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \perp\end{array}}}{\phi}\ \textit{PBC} \quad (\textit{or}\ \ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise
2	$\neg\phi$	assumption

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{matrix}\neg\phi \\ \vdots \\ \bot\end{matrix}}}{\phi}\ \textit{PBC}\quad (\textit{or}\ \ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg_e$ 2,1

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \textit{PBC}\quad(\textit{or}\ \ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\phi$	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg_e$ 2,1
4	ϕ	PBC 2–3

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\phi \vee \neg\phi} \quad \textit{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \;\vdash\; \neg p \vee q$:

Law of Excluded Middle

Law of Excluded Middle Rule

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Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\phi \vee \neg\phi} \quad \textit{LEM}$$

(The rule does not have premises.)

Show that $p \to q \vdash \neg p \vee q$:

1	$p \to q$	premise
2	$p \vee \neg p$	LEM

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\phi \vee \neg\phi}\ \textit{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\phi \lor \neg\phi} \quad \textit{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \lor q$:

1	$p \rightarrow q$	premise
2	$p \lor \neg p$	LEM
3	p	assumption

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\phi \lor \neg\phi} \; LEM$$

(The rule does not have premises.)

Show that $p \to q \vdash \neg p \lor q$:

1	$p \to q$	premise
2	$p \lor \neg p$	LEM
3	p	assumption
4	q	$\to_e$ 3,1

Law of Excluded Middle

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$$\frac{}{\phi \vee \neg\phi} \; \textit{LEM}$$

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Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow_e$ 3,1
5	$\neg p \vee q$	$\vee_{i_2}$ 4

Law of Excluded Middle

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$$\frac{}{\phi \vee \neg\phi} \; \textit{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow_e$ 3,1
5	$\neg p \vee q$	$\vee_{i_2}$ 4

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2	$p \lor \neg p$	LEM
3	p	assumption
4	q	$\to_e$ 3,1
5	$\neg p \lor q$	$\lor_{i_2}$ 4
6	$\neg p$	assumption

Law of Excluded Middle

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(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

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2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow_e$ 3,1
5	$\neg p \vee q$	$\vee_{i_2}$ 4
6	$\neg p$	assumption
7	$\neg p \vee q$	$\vee_{i_1}$ 7

Law of Excluded Middle

Law of Excluded Middle Rule

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(The rule does not have premises.)

Show that $p \to q \vdash \neg p \vee q$:

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2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\to_e$ 3,1
5	$\neg p \vee q$	$\vee_{i_2}$ 4
6	$\neg p$	assumption
7	$\neg p \vee q$	$\vee_{i_1}$ 7
8	$\neg p \vee q$	$\vee_e$ 2, 3–5, 6–7

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

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Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2
4	$\perp$	$\neg_e$ 3,1

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	$\neg\phi$	$\neg_i$ 2–4

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	$\neg\phi$	$\neg_i$ 2–4
6	$\phi \vee \neg\phi$	$\vee_{i_2}$ 5

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	$\neg\phi$	$\neg_i$ 2–4
6	$\phi \vee \neg\phi$	$\vee_{i_2}$ 5
7	$\bot$	$\neg_e$ 6,1

Law of Excluded Middle is Derivable

The LEM rule $\vdash \phi \vee \neg\phi$ is derivable:

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	$\neg\phi$	$\neg_i$ 2–4
6	$\phi \vee \neg\phi$	$\vee_{i_2}$ 5
7	$\bot$	$\neg_e$ 6,1
8	$\phi \vee \neg\phi$	PBC 1–7

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

Example from a Previous Exam

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1	$q \vee \neg q$	LEM

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Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5
7	$\neg q$	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5
7	$\neg q$	assumption
8	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 6

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5
7	$\neg q$	assumption
8	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 6
9	$\neg q \vee (p \rightarrow q)$	$\vee_e$ 1, 2–6, 7–8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \to q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \to q))$	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \to q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \to q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \to q)$	$\vee_{i_1}$ 2

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \to q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \to q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \to q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\perp$	$\neg_e$ 3,1
5	q	PBC 2–4

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \to q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \to q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \to q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\perp$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	$\rightarrow_i$ 6–7

Example from a Previous Exam

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1	$\neg(\neg q \vee (p \to q))$	assumption
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4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \to q$	$\to_i$ 6–7
9	$\neg q \vee (p \to q)$	$\vee_{i_2}$ 8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	$\rightarrow_i$ 6–7
9	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 8
10	$\bot$	$\neg_e$ 9,1

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\perp$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	$\rightarrow_i$ 6–7
9	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 8
10	$\perp$	$\neg_e$ 9,1
11	$\neg q \vee (p \rightarrow q)$	PBC 1–11

More Exam Preparation Tasks

Exam Exercises

Try to derive yourself:

- $p \vee q,\ \neg p \ \vdash\ q$
- $p \rightarrow (q \rightarrow r) \ \vdash\ q \rightarrow (p \rightarrow r)$
- $(p \rightarrow q) \rightarrow r \ \vdash\ p \rightarrow (q \rightarrow r)$
- $p \vee (q \wedge r) \ \vdash\ p \vee q$
- $a \vee b,\ a \rightarrow c,\ \neg d \rightarrow \neg b \ \vdash\ c \vee d$
- $(a \rightarrow b) \wedge (b \rightarrow a) \ \vdash\ (a \wedge b) \vee (\neg a \wedge \neg b)$
- $a \wedge (b \vee c) \ \vdash\ (a \wedge b) \vee (a \wedge c)$
- $\vdash\ ((p \rightarrow q) \rightarrow p) \rightarrow p$