

Logic and Modelling

— Natural Deduction for Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Natural Deduction

$$\alpha_1, \dots, \alpha_n \vdash \beta$$

means: there exists a **natural deduction derivation** with

- ▶ premises $\alpha_1, \dots, \alpha_n$, and
- ▶ conclusion β .

Natural Deduction

$$\alpha_1, \dots, \alpha_n \vdash \beta$$

means: there exists a **natural deduction derivation** with

- ▶ premises $\alpha_1, \dots, \alpha_n$, and
- ▶ conclusion β .

Natural deduction is a **formal system** with **strict formal rules!**

Rules for \wedge and \vee

Introduction of \wedge

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \wedge_i$$

(If you have derived α and β , then you can conclude $\alpha \wedge \beta$.)

Rules for \wedge and \vee

Introduction of \wedge

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \wedge_i$$

(If you have derived α and β , then you can conclude $\alpha \wedge \beta$.)

Elimination of \wedge

$$\frac{\alpha \wedge \beta}{\alpha} \wedge_{e_1}$$

$$\frac{\alpha \wedge \beta}{\beta} \wedge_{e_2}$$

Rules for \wedge and \vee

Introduction of \wedge

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \wedge_i$$

(If you have derived α and β , then you can conclude $\alpha \wedge \beta$.)

Elimination of \wedge

$$\frac{\alpha \wedge \beta}{\alpha} \wedge_{e_1}$$

$$\frac{\alpha \wedge \beta}{\beta} \wedge_{e_2}$$

Rules for \vee

$$\frac{\alpha}{\alpha \vee \beta} \vee_{i_1}$$

$$\frac{\beta}{\alpha \vee \beta} \vee_{i_2}$$

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1 $(p \wedge q) \wedge \neg r$ premise

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1 $(p \wedge q) \wedge \neg r$ premise

2 $p \wedge q$ \wedge_{e_1} 1

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1	$(p \wedge q) \wedge \neg r$	premise
2	$p \wedge q$	\wedge_{e_1} 1
3	q	\wedge_{e_2} 2

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1	$(p \wedge q) \wedge \neg r$	premise
2	$p \wedge q$	\wedge_{e_1} 1
3	q	\wedge_{e_2} 2
4	$\neg r$	\wedge_{e_2} 1

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1	$(p \wedge q) \wedge \neg r$	premise
2	$p \wedge q$	\wedge_{e_1} 1
3	q	\wedge_{e_2} 2
4	$\neg r$	\wedge_{e_2} 1
5	$q \wedge \neg r$	\wedge_i 3, 4

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \wedge \neg r$ from $(p \wedge q) \wedge \neg r$?

1	$(p \wedge q) \wedge \neg r$	premise
2	$p \wedge q$	\wedge_{e_1} 1
3	q	\wedge_{e_2} 2
4	$\neg r$	\wedge_{e_2} 1
5	$q \wedge \neg r$	\wedge_i 3, 4

Hence we have derived

$$(p \wedge q) \wedge \neg r \vdash q \wedge \neg r$$

Rules for $\neg\neg$ and \rightarrow

Rules for $\neg\neg$

$$\frac{\neg\neg\alpha}{\alpha} \quad \neg\neg_e$$

$$\frac{\alpha}{\neg\neg\alpha} \quad \neg\neg_i$$

Rules for $\neg\neg$ and \rightarrow

Rules for $\neg\neg$

$$\frac{\neg\neg\alpha}{\alpha} \quad \neg\neg_e$$

$$\frac{\alpha}{\neg\neg\alpha} \quad \neg\neg_i$$

Elimination rules for \rightarrow

This rule is called “Modus Ponens” (MP):

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta} \quad \rightarrow_e \text{ (or } MP)$$

Rules for $\neg\neg$ and \rightarrow

Rules for $\neg\neg$

$$\frac{\neg\neg\alpha}{\alpha} \quad \neg\neg_e$$

$$\frac{\alpha}{\neg\neg\alpha} \quad \neg\neg_i$$

Elimination rules for \rightarrow

This rule is called “Modus Ponens” (MP):

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta} \quad \rightarrow_e \text{ (or } MP)$$

This rule is called “Modus Tollens” (MT):

$$\frac{\alpha \rightarrow \beta \quad \neg\beta}{\neg\alpha} \quad MT$$

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r$?

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

4 $\neg\neg p$ $\neg\neg_i$ 2

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

4 $\neg\neg p$ $\neg\neg_i$ 2

5 $\neg q \rightarrow r$ \rightarrow_e 4,1

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r$?

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \rightarrow r$	\rightarrow_e 4,1
6	$\neg\neg q$	MT 5,3

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r$?

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

4 $\neg\neg p$ $\neg\neg_i$ 2

5 $\neg q \rightarrow r$ \rightarrow_e 4,1

6 $\neg\neg q$ MT 5,3

7 q $\neg\neg_e$ 6

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r$?

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \rightarrow r$	\rightarrow_e 4,1
6	$\neg\neg q$	MT 5,3
7	q	$\neg\neg_e$ 6

Hence we have derived

$$\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$$

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{|c|} \hline \alpha \\ \vdots \\ \beta \\ \hline \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1 $p \rightarrow q$ premise

Introduction of \rightarrow

Introduction rule for \rightarrow

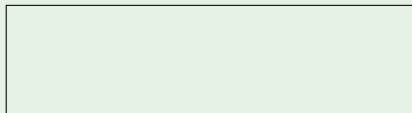
$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1

$p \rightarrow q$

premise



Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2

Introduction of \rightarrow

Introduction rule for \rightarrow

$$\frac{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow_i$$

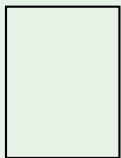
Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	\rightarrow_i 2-3

Block Structures

Allowed block structures

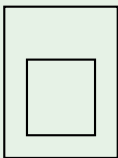
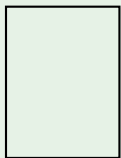
Blocks are allowed to be nested inside each other:



Block Structures

Allowed block structures

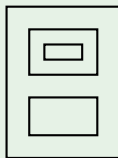
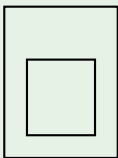
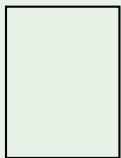
Blocks are allowed to be nested inside each other:



Block Structures

Allowed block structures

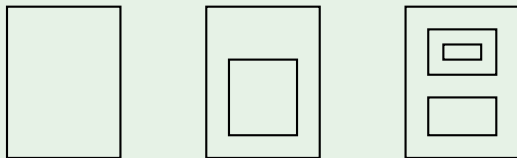
Blocks are allowed to be nested inside each other:



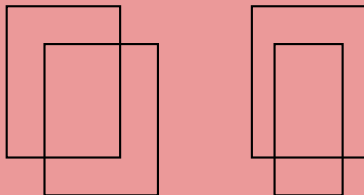
Block Structures

Allowed block structures

Blocks are allowed to be nested inside each other:



Blocks are not allowed to intersect:



Block Structure

When applying a rule

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta},$$

the $\alpha_1, \dots, \alpha_n$ must be **in the scope**, that is, must have been derived in the current block or a surrounding block.

(Compare with scopes of variables in programming languages.)

Block Structure

When applying a rule

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta},$$

the $\alpha_1, \dots, \alpha_n$ must be **in the scope**, that is, must have been derived in the current block or a surrounding block.

(Compare with scopes of variables in programming languages.)

1	$p \rightarrow q$	premise
2	p	assumption
3	q	\rightarrow_e 1,2
4	$q \vee q$	\vee_i 3 This is not allowed!!!

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1 $p \rightarrow (q \rightarrow r)$ premise

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1 $p \rightarrow (q \rightarrow r)$ premise

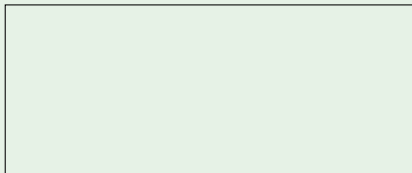
2 p premise

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1 $p \rightarrow (q \rightarrow r)$ premise

2 p premise



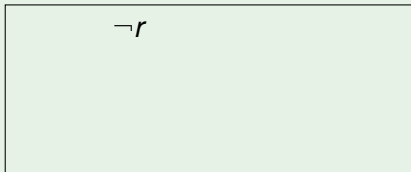
Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1 $p \rightarrow (q \rightarrow r)$ premise

2 p premise

3 $\neg r$ assumption



Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	\rightarrow_e 2,1

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	\rightarrow_e 2,1
5	$\neg q$	MT 4,3

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	\rightarrow_e 2,1
5	$\neg q$	MT 4,3
6	$\neg r \rightarrow \neg q$	\rightarrow_i 3-5

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r), p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	\rightarrow_e 2,1
5	$\neg q$	MT 4,3
6	$\neg r \rightarrow \neg q$	\rightarrow_i 3-5

Hence we have derived

$$p \rightarrow (q \rightarrow r), p \vdash \neg r \rightarrow \neg q$$

Special Cases

1	p	assumption
2	$p \rightarrow p$	\rightarrow_i 1-1

This is a derivation of

$$\vdash p \rightarrow p$$

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

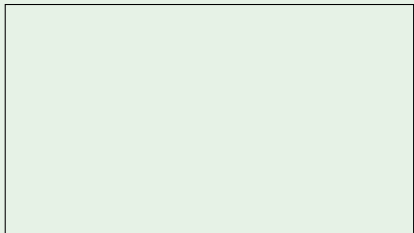
Lets try to prove $\vdash p \rightarrow (q \rightarrow p) \quad !$

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!



Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1

p

assumption

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1

p

assumption



Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1

p

assumption

2

q

assumption

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1 p assumption

2 q assumption

3 p copy 1

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	\rightarrow_i 2-3

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	\rightarrow_i 2-3
5	$p \rightarrow (q \rightarrow p)$	\rightarrow_i 1-4

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \text{ copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	\rightarrow_i 2-3
5	$p \rightarrow (q \rightarrow p)$	\rightarrow_i 1-4

This concludes the derivation.

Rules for \neg and \perp

Rules for \neg and \perp

$$\frac{\alpha \quad \neg\alpha}{\perp} \neg_e$$

Rules for \neg and \perp

Rules for \neg and \perp

$$\frac{\alpha \quad \neg\alpha}{\perp} \neg_e$$

$$\frac{\boxed{\begin{array}{c} \alpha \\ \vdots \\ \perp \end{array}}}{\neg\alpha} \neg_i$$

Rules for \neg and \perp

Rules for \neg and \perp

$$\frac{\alpha \quad \neg\alpha}{\perp} \neg_e$$

$$\frac{\boxed{\begin{array}{c} \alpha \\ \vdots \\ \perp \end{array}}}{\neg\alpha} \neg_i$$

$$\frac{\perp}{\alpha} \perp_e$$

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1 $\neg\neg p \rightarrow (\neg q \rightarrow r)$ premise

2 p premise

3 $\neg r$ premise

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5

Example

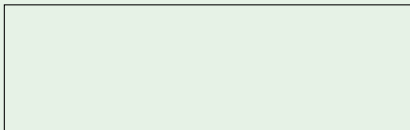
Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1



Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption
9	r	\rightarrow_e 8,7

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r)$, p , $\neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption
9	r	\rightarrow_e 8,7
10	\perp	\neg_e 9,3

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption
9	r	\rightarrow_e 8,7
10	\perp	\neg_e 9,3
11	$\neg\neg q$	\neg_i 8–10

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r), p, \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	\perp	\neg_e 2,4
6	$\neg\neg p$	\neg_i 4–5
7	$\neg q \rightarrow r$	\rightarrow_e 6,1
8	$\neg q$	assumption
9	r	\rightarrow_e 8,7
10	\perp	\neg_e 9,3
11	$\neg\neg q$	\neg_i 8–10
12	q	$\neg\neg_e$ 11

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

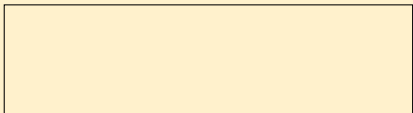
1	α	premise
---	----------	---------

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

1 α premise



MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

1	α	premise
2	$\neg\alpha$	assumption

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

1	α	premise
2	$\neg\alpha$	assumption
3	\perp	\neg_e 1,2

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

1	α	premise
2	$\neg\alpha$	assumption
3	\perp	\neg_e 1,2
4	$\neg\neg\alpha$	\neg_i 2-3

MT and $\neg\neg_i$ as “derived rules”

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α .

We can derive it using other rules as follows:

1	α	premise
2	$\neg\alpha$	assumption
3	\perp	\neg_e 1,2
4	$\neg\neg\alpha$	\neg_i 2–3

Thus: $\alpha \vdash \neg\neg\alpha$.

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1 $\alpha \rightarrow \beta$ premise

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

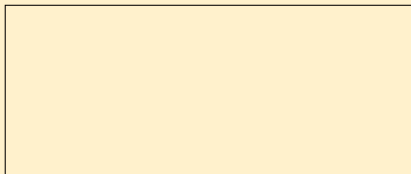
- | | | |
|---|----------------------------|---------|
| 1 | $\alpha \rightarrow \beta$ | premise |
| 2 | $\neg\beta$ | premise |

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

- | | | |
|---|----------------------------|---------|
| 1 | $\alpha \rightarrow \beta$ | premise |
| 2 | $\neg\beta$ | premise |



MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

- | | | |
|---|---|------------|
| 1 | $\alpha \rightarrow \beta$ | premise |
| 2 | $\neg\beta$ | premise |
| 3 | <div style="border: 1px solid black; display: inline-block; width: 400px; height: 150px; vertical-align: middle;"></div> α | assumption |

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \rightarrow \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	\rightarrow_e 3,1

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \rightarrow \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	\rightarrow_e 3,1
5	\perp	\neg_e 2,4

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \rightarrow \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	\rightarrow_e 3,1
5	\perp	\neg_e 2,4
6	$\neg\alpha$	\neg_i 3–5

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \rightarrow \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	\rightarrow_e 3,1
5	\perp	\neg_e 2,4
6	$\neg\alpha$	\neg_i 3–5

Thus: $\alpha \rightarrow \beta, \neg\beta \vdash \neg\alpha$.

MT and $\neg\neg_i$ as “derived rules”

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \rightarrow \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \rightarrow \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	\rightarrow_e 3,1
5	\perp	\neg_e 2,4
6	$\neg\alpha$	\neg_i 3–5

Thus: $\alpha \rightarrow \beta, \neg\beta \vdash \neg\alpha$.

This shows that $\neg\neg_i$ and MT are **not needed**, but sometimes help to make derivations easier or shorter.

Elimination of \vee

Elimination of \vee

$$\frac{\alpha \vee \beta \quad \begin{array}{|c|} \alpha \\ \vdots \\ \gamma \end{array} \quad \begin{array}{|c|} \beta \\ \vdots \\ \gamma \end{array}}{\gamma} \vee_e$$

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1 $p \vee \neg q$ premise

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1 $p \vee \neg q$ premise

2 $\neg p \rightarrow q$ premise

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1 $p \vee \neg q$ premise

2 $\neg p \rightarrow q$ premise

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

- | | | |
|---|---|------------|
| 1 | $p \vee \neg q$ | premise |
| 2 | $\neg p \rightarrow q$ | premise |
| 3 | <div style="border: 1px solid black; padding: 5px; display: inline-block;">p</div> | assumption |

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5

Example

Prove $p \vee \neg q, \neg p \rightarrow q \vdash p$!

1	$p \vee \neg q$	premise
2	$\neg p \rightarrow q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5
7	p	\vee_e 1, 3-3, 4-6

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1 $\neg p \vee q$ premise

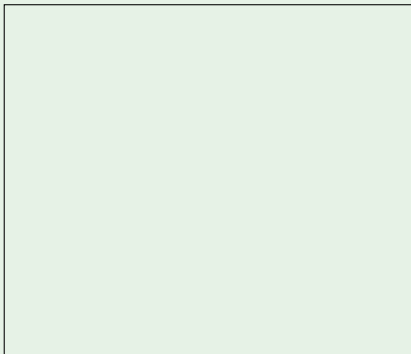
Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1

$\neg p \vee q$

premise

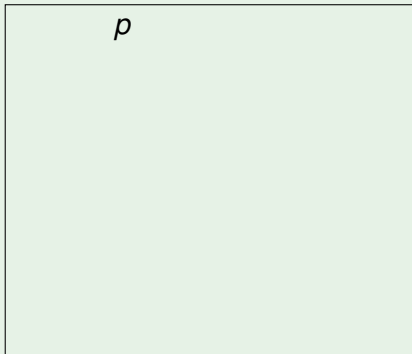


Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1 $\neg p \vee q$ premise

2 p assumption

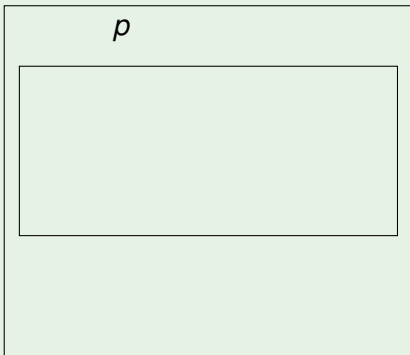


Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1 $\neg p \vee q$ premise

2 p assumption



Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4
6	q	assumption

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4
6	q	assumption
7	q	\vee_e 1, 3–5, 6–6

Example

Use \perp_e to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	\perp	\neg_e 2,3
5	q	\perp_e 4
6	q	assumption
7	q	\vee_e 1, 3–5, 6–6
8	$p \rightarrow q$	\rightarrow_i 2–7

Proof by Contradiction

Assume that you have derived

$\neg\alpha$

\vdots

\perp

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}$$

Then also

$$\begin{array}{c} \boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}} \\ \neg\neg\alpha \quad \neg_i \\ \alpha \quad \neg\neg_e \end{array}$$

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}$$

Then also

$$\begin{array}{c} \boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}} \\ \neg\neg\alpha \quad \neg_i \\ \alpha \quad \neg\neg_e \end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}$$

Then also

$$\begin{array}{c} \boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}} \\ \neg\neg\alpha \quad \neg_i \\ \alpha \quad \neg\neg_e \end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Also known as **Reductio ad Absurdum (RAA)**!

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}}}{\alpha} \quad \text{PBC (or RAA)}$$

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}}}{\alpha} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

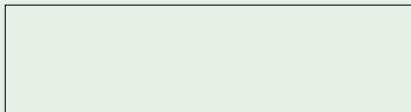
Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\begin{array}{|c} \neg\alpha \\ \vdots \\ \perp \end{array}}{\alpha} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

1 $\neg\neg\alpha$ premise



Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\begin{array}{|c|} \hline \neg\alpha \\ \vdots \\ \perp \\ \hline \end{array}}{\alpha} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\alpha$	premise
2	$\neg\alpha$	assumption

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}}}{\alpha} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\alpha$	premise
2	$\neg\alpha$	assumption
3	\perp	\neg_e 2,1

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c} \neg\alpha \\ \vdots \\ \perp \end{array}}}{\alpha} \quad \text{PBC (or RAA)}$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\alpha$	premise
2	$\neg\alpha$	assumption
3	\perp	\neg_e 2,1
4	α	PBC 2–3

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1 $p \rightarrow q$ premise

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM

Law of Excluded Middle

Law of Excluded Middle Rule

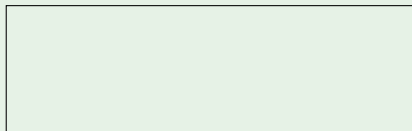
$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1 $p \rightarrow q$ premise

2 $p \vee \neg p$ LEM



Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	<div style="border: 1px solid black; display: inline-block; width: 150px; height: 50px; vertical-align: middle;"></div> p	assumption

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1

Law of Excluded Middle

Law of Excluded Middle Rule

$$\overline{\alpha \vee \neg \alpha} \quad \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4

Law of Excluded Middle

Law of Excluded Middle Rule

$$\overline{\alpha \vee \neg \alpha} \quad \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4
6	$\neg p$	assumption

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4
6	$\neg p$	assumption
7	$\neg p \vee q$	\vee_{i_1} 7

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg \alpha} \text{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

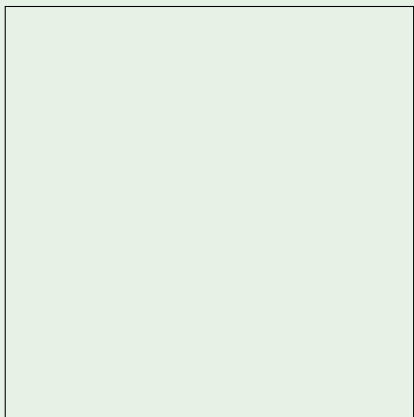
1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	\rightarrow_e 3,1
5	$\neg p \vee q$	\vee_{i_2} 4
6	$\neg p$	assumption
7	$\neg p \vee q$	\vee_{i_1} 7
8	$\neg p \vee q$	\vee_e 2, 3–5, 6–7

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:



Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1

$\neg(\alpha \vee \neg\alpha)$

assumption

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1

$\neg(\alpha \vee \neg\alpha)$

assumption



Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1

$\neg(\alpha \vee \neg\alpha)$

assumption

2

α

assumption

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	$\vee_{i_1} 2$

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	\vee_{i_1} 2
4	\perp	\neg_e 3,1

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	$\neg\alpha$	\neg_i 2-4

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	$\neg\alpha$	\neg_i 2-4
6	$\alpha \vee \neg\alpha$	\vee_{i_2} 5

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	$\neg\alpha$	\neg_i 2-4
6	$\alpha \vee \neg\alpha$	\vee_{i_2} 5
7	\perp	\neg_e 6,1

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	$\neg\alpha$	\neg_i 2-4
6	$\alpha \vee \neg\alpha$	\vee_{i_2} 5
7	\perp	\neg_e 6,1
8	$\alpha \vee \neg\alpha$	PBC 1-7

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1

$q \vee \neg q$

LEM

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1

$q \vee \neg q$

LEM



Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1

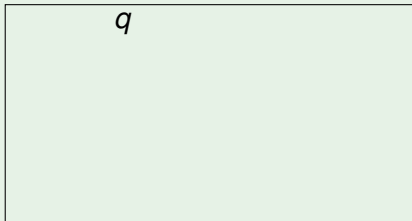
$q \vee \neg q$

LEM

2

q

assumption



Example from a Previous Exam

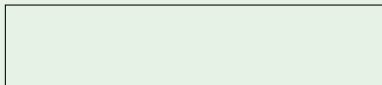
Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

LEM

2 q

assumption



Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

2 q

3 p

LEM

assumption

assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

2 q

3 p

4 q

LEM

assumption

assumption

copy 2

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

2 q

3 p

4 q

5 $p \rightarrow q$

LEM

assumption

assumption

copy 2

\rightarrow_i 3–4

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

LEM

2 q

assumption

3 p

assumption

4 q

copy 2

5 $p \rightarrow q$

\rightarrow_i 3–4

6 $\neg q \vee (p \rightarrow q)$

\vee_{i_2} 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

2 q

3 p

4 q

5 $p \rightarrow q$

6 $\neg q \vee (p \rightarrow q)$

LEM

assumption

assumption

copy 2

\rightarrow_i 3–4

\vee_{i_2} 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

2 q

3 p

4 q

5 $p \rightarrow q$

6 $\neg q \vee (p \rightarrow q)$

7 $\neg q$

LEM

assumption

assumption

copy 2

\rightarrow_i 3–4

\vee_{i_2} 5

assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

LEM

2 q

assumption

3 p

assumption

4 q

copy 2

5 $p \rightarrow q$

\rightarrow_i 3–4

6 $\neg q \vee (p \rightarrow q)$

\vee_{i_2} 5

7 $\neg q$

assumption

8 $\neg q \vee (p \rightarrow q)$

\vee_{i_1} 6

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1 $q \vee \neg q$

LEM

2 q

assumption

3 p

assumption

4 q

copy 2

5 $p \rightarrow q$

\rightarrow_i 3–4

6 $\neg q \vee (p \rightarrow q)$

\vee_{i_2} 5

7 $\neg q$

assumption

8 $\neg q \vee (p \rightarrow q)$

\vee_{i_1} 6

9 $\neg q \vee (p \rightarrow q)$

\vee_e 1, 2–6, 7–8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:



Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1

$$\neg(\neg q \vee (p \rightarrow q))$$

assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1

$\neg(\neg q \vee (p \rightarrow q))$

assumption



Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1

$\neg(\neg q \vee (p \rightarrow q))$

assumption

2

$\neg q$

assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1

Example from a Previous Exam

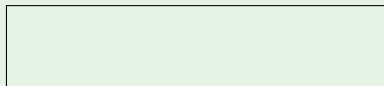
Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4



Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2-4
6	p	assumption

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	\rightarrow_i 6–7

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_{i_1} 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	\rightarrow_i 6–7
9	$\neg q \vee (p \rightarrow q)$	\vee_{i_2} 8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	\rightarrow_i 6–7
9	$\neg q \vee (p \rightarrow q)$	\vee_i 8
10	\perp	\neg_e 9,1

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	\vee_i 2
4	\perp	\neg_e 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	\rightarrow_i 6–7
9	$\neg q \vee (p \rightarrow q)$	\vee_i 8
10	\perp	\neg_e 9,1
11	$\neg q \vee (p \rightarrow q)$	PBC 1–11

More Exam Preparation Tasks

Exam Exercises

Try to derive yourself:

- ▶ $p \vee q, \neg p \vdash q$
- ▶ $p \rightarrow (q \rightarrow r) \vdash q \rightarrow (p \rightarrow r)$
- ▶ $(p \rightarrow q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$
- ▶ $p \vee (q \wedge r) \vdash p \vee q$
- ▶ $a \vee b, a \rightarrow c, \neg d \rightarrow \neg b \vdash c \vee d$
- ▶ $(a \rightarrow b) \wedge (b \rightarrow a) \vdash (a \wedge b) \vee (\neg a \wedge \neg b)$
- ▶ $a \wedge (b \vee c) \vdash (a \wedge b) \vee (a \wedge c)$
- ▶ $\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$