Logic and Modelling

— Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Propositional Logic: Introduction

Propositions are “declarative sentences”.

Examples

It rains.

$8 > 5$.

$6 < 2$.

X is positive.

The earth is flat.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p &: \quad \text{It rains.} \\ q &: \quad \text{Janneke has her umbrella.} \\ r &: \quad \text{Janneke is wet.} \end{aligned}$$

Then the sentences above becomes:

$$(p \wedge \neg q) \rightarrow r, \;\; \neg r, \;\; p \quad \models \quad q$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p \qquad\qquad p \wedge q$$
$$\neg p \qquad\qquad p \vee \neg q$$
$$(\neg p \wedge q) \rightarrow r$$

We use Greek letters for formulas:

ϕ	phi	ψ	psi	ξ	xi
α	alpha	β	beta		$\ldots$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

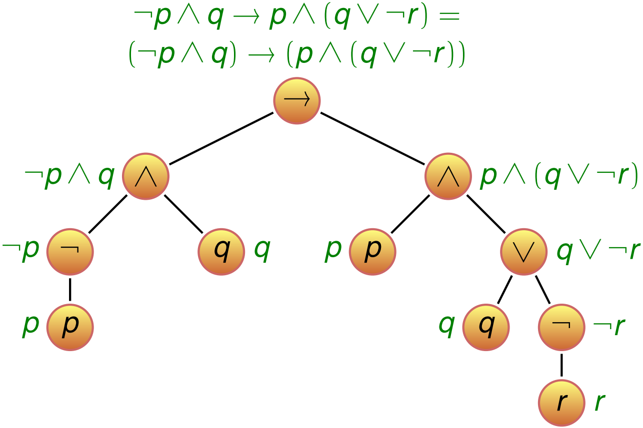
$p \wedge q \vee r \quad =$ **unclear! needs brackets!**

$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)}$$

$p \rightarrow q \rightarrow r \quad =$ **unclear! needs brackets! (not associative)**

Parse Trees

What is the formula corresponding to the following parse tree?



Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \rightarrow p$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

Contradiction = always false

$$p \wedge \neg p$$

$$((p \rightarrow q) \rightarrow p) \wedge \neg p$$

Properties of Formulas

Equivalent formulas = true at the same time

$$\neg p \vee q \quad \equiv \quad p \rightarrow q$$

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

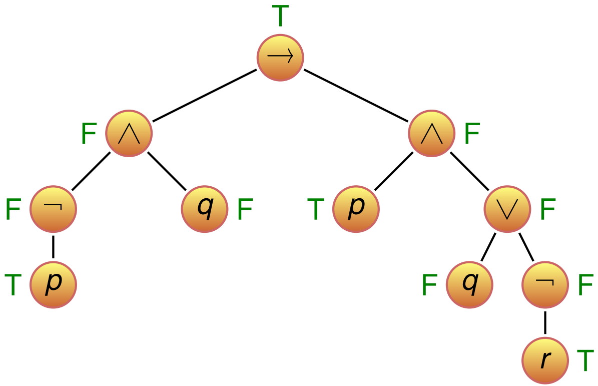
$\bot$
F

$\top$
T

Parse Trees

Bottom-up evaluation of truth values in a parse tree:

- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

contradiction: always F

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \rightarrow q$$

More examples

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$

$$p \vee q \equiv q \vee p$$

$$p \rightarrow \neg q \equiv q \rightarrow \neg p$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\[1ex]
\phi \wedge \phi &\equiv \phi \\
\phi \vee \phi &\equiv \phi \\[1ex]
\phi \wedge \psi &\equiv \psi \wedge \phi \\
\phi \vee \psi &\equiv \psi \vee \phi \\[1ex]
\phi \rightarrow \psi &\equiv \neg\phi \vee \psi \\
\phi \rightarrow \psi &\equiv \neg\psi \rightarrow \neg\phi
\end{aligned}
$$

De Morgan laws

$$
\begin{aligned}
\neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi \\
\neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi
\end{aligned}
$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ follows ψ.

This is denoted as:

$$\phi_1, \ldots, \phi_n \;\therefore\; \psi$$

$$p \land \neg q \;\therefore\; \neg q$$

$$p \to (q \to r),\; p,\; \neg r \;\therefore\; \neg q$$

Two Important Variants

semantic	$\phi_1, \ldots, \phi_n \models \psi$
syntactic	$\phi_1, \ldots, \phi_n \vdash \psi$

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \rightarrow q$ is T. Hence: $q \models p \rightarrow q$.

Examples Semantic Implication

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \rightarrow q$ and $\neg q$ are T

In this line(s) $\neg p$ is true.

Hence $p \rightarrow q, \neg q \models \neg p$ holds.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $\quad p \to q, \neg q \models \neg p \quad$?

Assume that $p \to q$ and $\neg q$ are T.

Then q is F.

Then p must be F since otherwise $p \to q$ was F.

Thus $\neg p$ is T.

Hence $p \to q, \neg q \models \neg p$ holds.

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $p \vee q \models p \rightarrow q$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

Conclusion: $p \vee q \not\models p \rightarrow q$.

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q, q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \rightarrow q$?
- $p \rightarrow (q \rightarrow r) \models q \rightarrow (p \rightarrow r)$?
- $p \rightarrow (q \rightarrow r) \models (p \rightarrow q) \rightarrow r$?
- $(p \rightarrow q) \rightarrow r \models p \rightarrow (q \rightarrow r)$?

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \Longleftrightarrow \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad \Longleftrightarrow \quad \phi \models \psi \text{ and } \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Note that $\equiv$ is an equivalence relation:

- reflexive,
- symmetric,
- transitive.

Other Important Facts

$$\phi \models \psi \iff \models \phi \to \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \iff \phi_1, \ldots, \phi_{n-1} \models \phi_n \to \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \iff \models \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$$

$$\models \phi \land \psi \iff \models \phi \text{ and } \models \psi$$

$$\models \phi \text{ or } \models \psi \implies \models \phi \lor \psi$$

$$\models \phi \lor \psi \not\Longrightarrow \models \phi \text{ or } \models \psi$$

For example: $\models p \lor \neg p$, but not $\models p$ and not $\models \neg p$.