Logic and Modelling

— Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Propositional Logic: Introduction

Propositions are “declarative sentences”.

Examples

It rains.

$8 > 5$.

$6 < 2$.

X is positive.

The earth is flat.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

r : It rains.

u : Janneke has her umbrella.

w : Janneke is wet.

Then the sentences above becomes:

$$(r \wedge \neg u) \to w, \;\; \neg w, \;\; r \quad \models \quad u$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p \qquad p \wedge q$$
$$\neg p \qquad p \vee \neg q$$
$$(\neg p \wedge q) \rightarrow r$$

We use Greek letters for formulas:

ϕ	phi	ψ	psi	ξ	xi
α	alpha	β	beta	$\ldots$	

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\wedge, \vee$	binding-strength is in-between
$\rightarrow$	binds the weakest

How do we read the following?

$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$

$\neg p \wedge r \quad = \quad (\neg p) \wedge r$

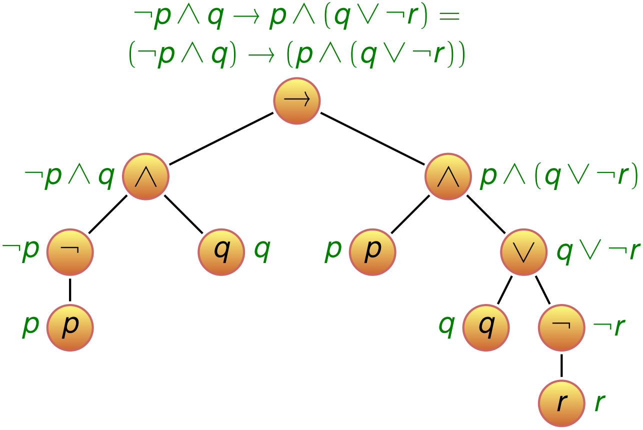
$p \wedge q \vee r \quad =$ **unclear! needs brackets!**

$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r)$ (associativity)

$p \rightarrow q \rightarrow r \quad =$ **unclear! needs brackets! (not associative)**

Parse Trees

What is the formula corresponding to the following parse tree?



Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \to p$$

$$((p \to q) \to p) \to p$$

Contradiction = always false

$$p \wedge \neg p$$

$$((p \to q) \to p) \wedge \neg p$$

Properties of Formulas

Equivalent formulas = true at the same time

$$\neg p \vee q \;\equiv\; p \rightarrow q$$

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

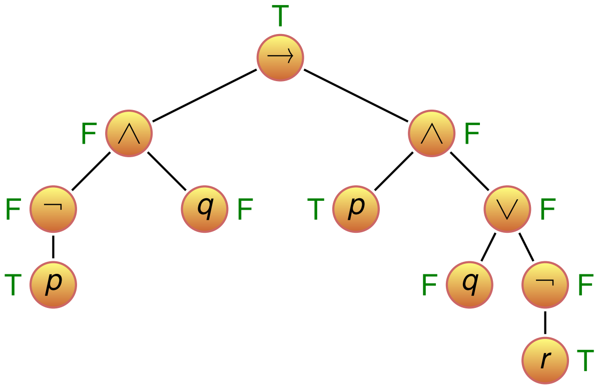
$\perp$
F

$\top$
T

Parse Trees

Bottom-up evaluation of truth values in a parse tree:

- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

contradiction: always F

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi\,,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \to q$$

More examples

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$
$$p \vee q \equiv q \vee p$$
$$p \to \neg q \equiv q \to \neg p$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\[1ex]
\alpha \wedge \beta &\equiv \beta \wedge \alpha \\
\alpha \vee \beta &\equiv \beta \vee \alpha \\[1ex]
\alpha \rightarrow \beta &\equiv \neg\alpha \vee \beta \\
\alpha \rightarrow \beta &\equiv \neg\beta \rightarrow \neg\alpha
\end{aligned}
$$

De Morgan laws

$$
\begin{aligned}
\neg(\alpha \vee \beta) &\equiv \neg\alpha \wedge \neg\beta \\
\neg(\alpha \wedge \beta) &\equiv \neg\alpha \vee \neg\beta
\end{aligned}
$$

Logical Reasoning

From $\alpha_1, \ldots, \alpha_n$ follows β.

This is denoted as:

$$\alpha_1, \ldots, \alpha_n \ \therefore\ \beta$$

$$p \wedge \neg q \ \therefore\ \neg q$$

$$p \rightarrow (q \rightarrow r),\ p,\ \neg r \ \therefore\ \neg q$$

Two Important Variants

semantic	$\alpha_1, \ldots, \alpha_n \models \beta$
syntactic	$\alpha_1, \ldots, \alpha_n \vdash \beta$

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \to q$?

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \to q$ is T. Hence: $q \models p \to q$.

Examples Semantic Entailment

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \rightarrow q$ and $\neg q$ are T

In this line(s) $\neg p$ is true.

Hence $p \rightarrow q, \neg q \models \neg p$ holds.

Examples Semantic Entailment

To show that $\alpha_1, \ldots, \alpha_n \models \beta$ we need to show

- β is true whenever $\alpha_1, \ldots, \alpha_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\alpha_1, \ldots, \alpha_n$ are true, and
- show that β must be true as well.

Do we have $p \to q, \neg q \models \neg p$?

Assume that $p \to q$ and $\neg q$ are T.

Then q is F.

Then p must be F since otherwise $p \to q$ was F.

Thus $\neg p$ is T.

Hence $p \to q, \neg q \models \neg p$ holds.

Disproving Semantic Entailment

How to disprove $\alpha_1, \ldots, \alpha_n \models \beta$? That is, $\alpha_1, \ldots, \alpha_n \not\models \beta$?

Find a valuation (assignment of truth values to variables) that

- makes $\alpha_1, \ldots, \alpha_n$ true, and
- β false.

Do we have $p \vee q \models p \rightarrow q$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

Conclusion: $p \vee q \not\models p \rightarrow q$.

Examples Semantic Entailment

Which of the following semantic entailments hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \rightarrow q$?
- $p \rightarrow (q \rightarrow r) \models q \rightarrow (p \rightarrow r)$?
- $p \rightarrow (q \rightarrow r) \models (p \rightarrow q) \rightarrow r$?
- $(p \rightarrow q) \rightarrow r \models p \rightarrow (q \rightarrow r)$?

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \Longleftrightarrow \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\alpha \equiv \beta \quad \Longleftrightarrow \quad \alpha \models \beta \;\text{ and }\; \beta \models \alpha$$

(In other words: α and β have the same truth table.)

Note that $\equiv$ is an equivalence relation:

- reflexive,
- symmetric,
- transitive.

Other Important Facts

$$\alpha \models \beta \quad \Longleftrightarrow \quad \models \alpha \rightarrow \beta$$

$$\alpha_1, \ldots, \alpha_n \models \beta \quad \Longleftrightarrow \quad \alpha_1, \ldots, \alpha_{n-1} \models \alpha_n \rightarrow \beta$$

$$\alpha_1, \ldots, \alpha_n \models \beta \quad \Longleftrightarrow \quad \models \alpha_1 \rightarrow (\alpha_2 \rightarrow (\ldots(\alpha_n \rightarrow \beta)\ldots))$$

$$\models \alpha \wedge \beta \quad \Longleftrightarrow \quad \models \alpha \text{ and } \models \beta$$

$$\models \alpha \text{ or } \models \beta \quad \Longrightarrow \quad \models \alpha \vee \beta$$

$$\models \alpha \vee \beta \quad \not\Longrightarrow \quad \models \alpha \text{ or } \models \beta$$

For example: $\models p \vee \neg p$, but not $\models p$ and not $\models \neg p$.