Logic and Modelling

— Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Propositional Logic: Introduction

Propositions are “declarative sentences”.

Examples

It rains.

$8 > 5$.

$6 < 2$.

X is positive.

The earth is flat.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella,
  then she is wet.

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We can translate this to propositional logic as follows:

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u : Janneke has her umbrella.

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$$

Then the sentences above becomes:

$$(r \wedge \neg u) \rightarrow w$$

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Propositional Logic: Syntax

Syntax of Propositional Logic

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- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

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- propositional variables

$$p, \; q, \; r, \; \ldots \quad p_0, \; p_1, \; \ldots$$

- propositional connectives

$$\neg, \; \wedge, \; \vee, \; \rightarrow$$

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Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p \qquad \neg p \qquad (\neg p \wedge q) \rightarrow r$$

$$p \wedge q \qquad p \vee \neg q$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p \qquad\qquad p \wedge q$$
$$\neg p \qquad\qquad p \vee \neg q$$
$$(\neg p \wedge q) \rightarrow r$$

We use Greek letters for formulas:

ϕ	phi	ψ	psi	ξ	xi
α	alpha	β	beta	$\ldots$	

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\wedge, \vee$	binding-strength is in-between
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p \wedge q \rightarrow r &= (p \wedge q) \rightarrow r \\
\neg p \wedge r &= (\neg p) \wedge r \\
p \wedge q \vee r &=
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p \wedge q \vee r \quad &= \quad \textbf{unclear! needs brackets!} \\
p \wedge q \wedge r \quad &=
\end{aligned}
$$

Bracket Convention

Priority Rules

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p \wedge q \vee r &= \textbf{unclear! needs brackets!} \\
p \wedge q \wedge r &= (p \wedge q) \wedge r = p \wedge (q \wedge r)
\end{aligned}
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$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)}$$

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$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)}$$

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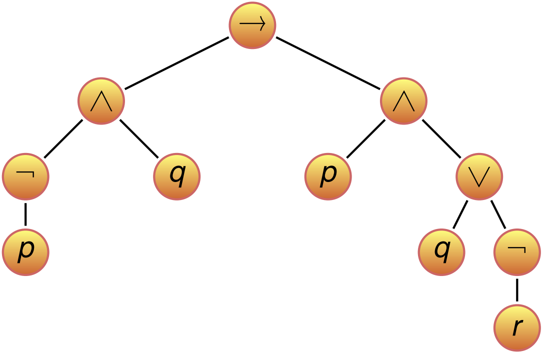
$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r)$ (associativity)

$p \rightarrow q \rightarrow r \quad =$ **unclear! needs brackets! (not associative)**

Parse Trees

What is the formula corresponding to the following parse tree?



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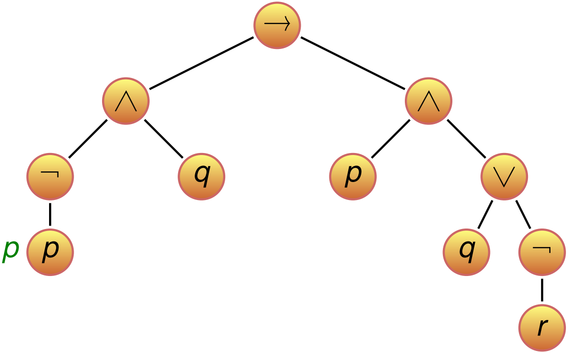

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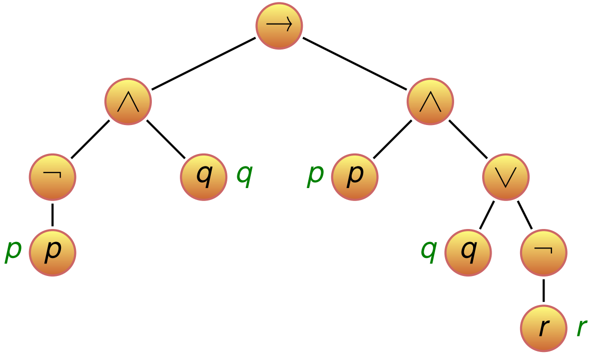

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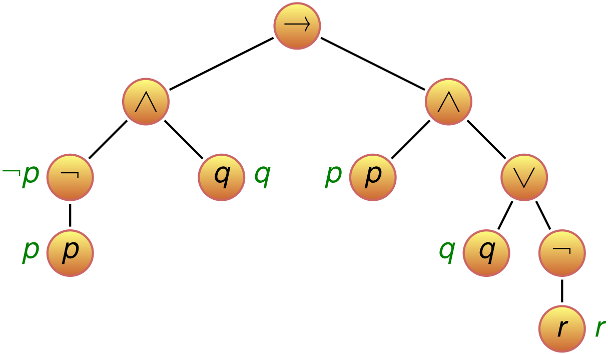

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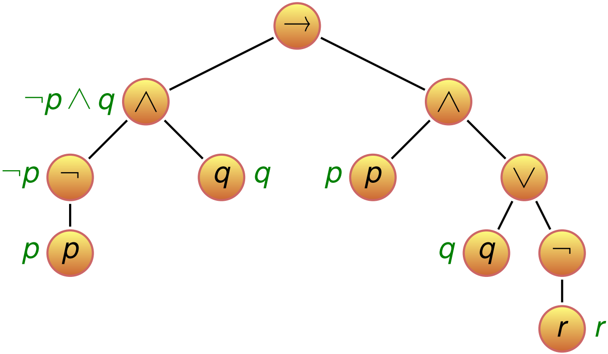

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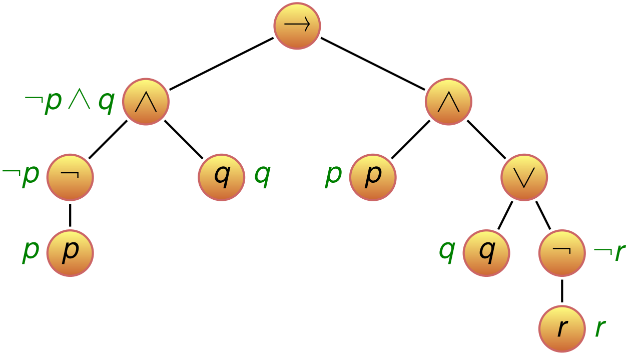

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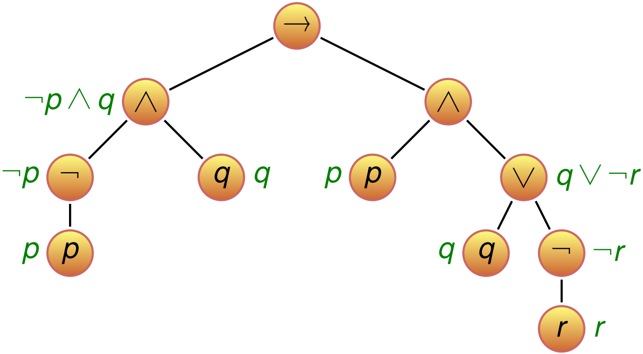

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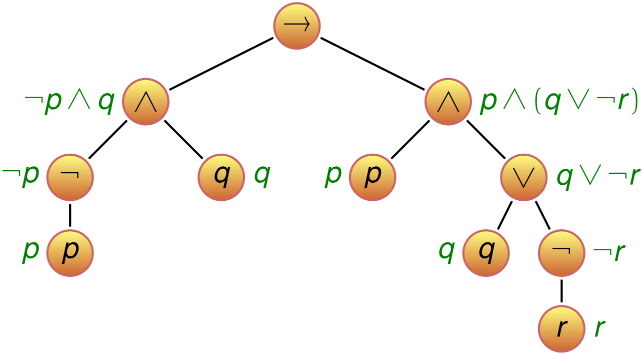

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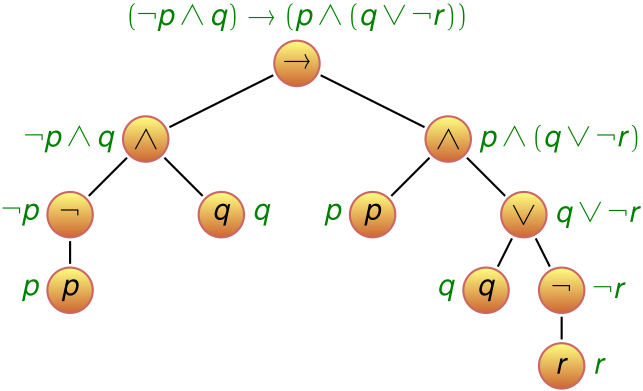

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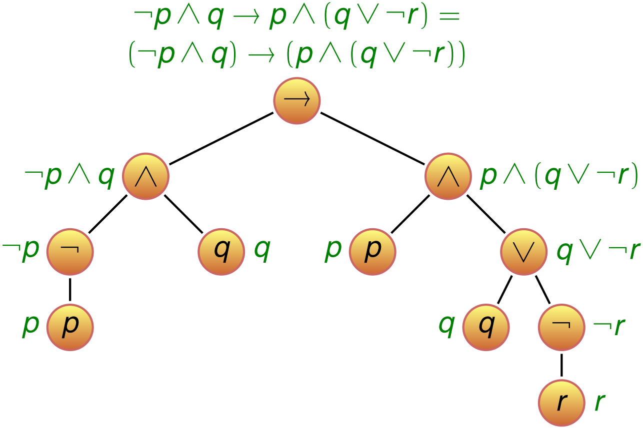

Properties of Formulas

Contingency

Tautology

Contradiction

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Contingency = sometimes true and sometimes false

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$$p \wedge q \wedge \neg r$$

Tautology

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$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \rightarrow p$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

Contradiction

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Properties of Formulas

Equivalent formulas

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Equivalent formulas = true at the same time

Properties of Formulas

Equivalent formulas = true at the same time

$$\neg p \lor q \;\equiv\; p \to q$$

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	
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T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	
F	T	
T	F	
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Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
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F	F	F
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$\perp$
F

Truth Tables

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F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

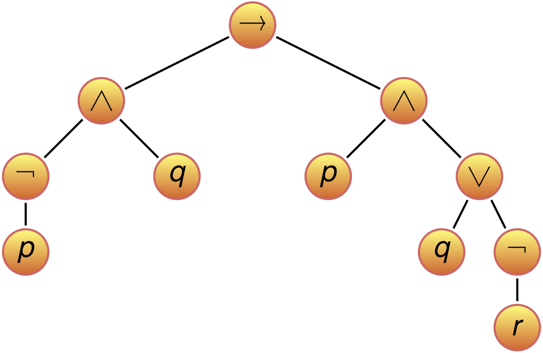
$\bot$
F

$\top$
T

Parse Trees

Bottom-up evaluation of truth values in a parse tree:

- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



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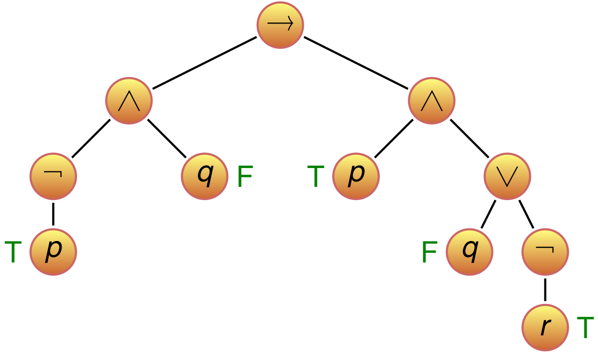

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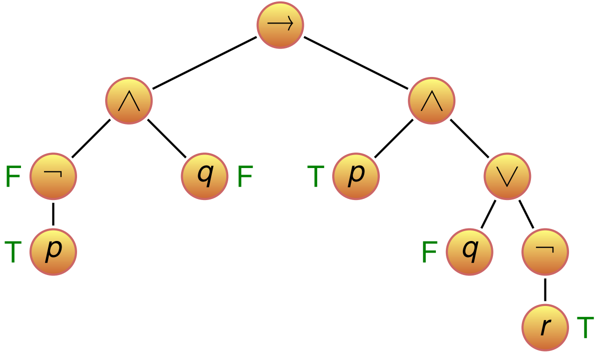

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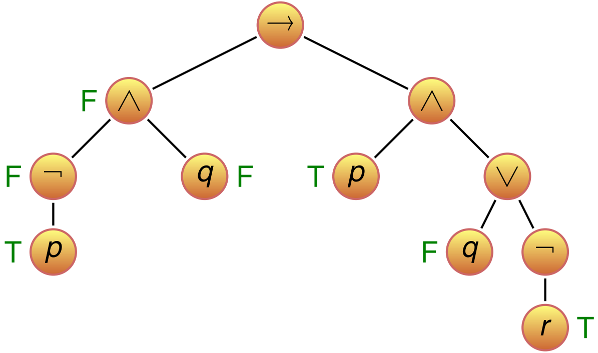

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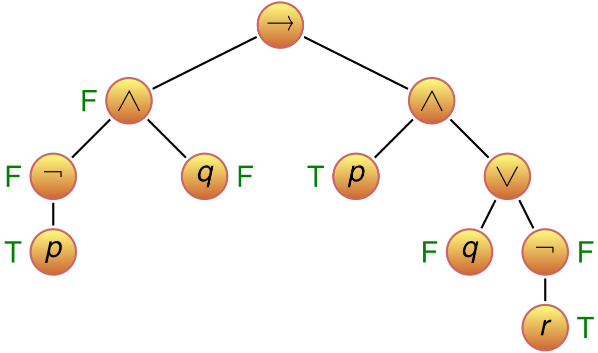

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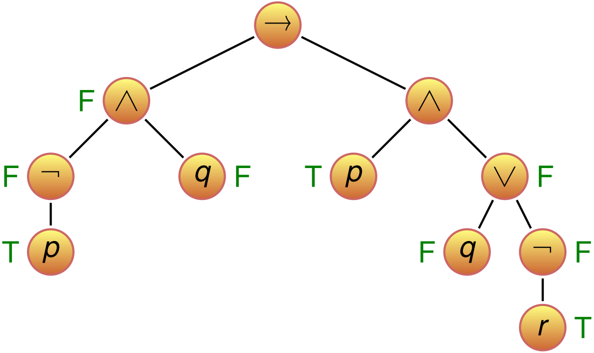

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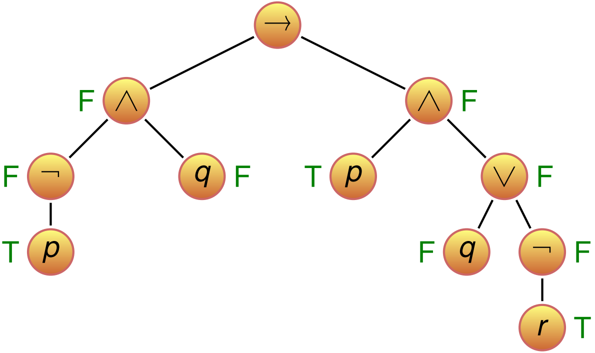

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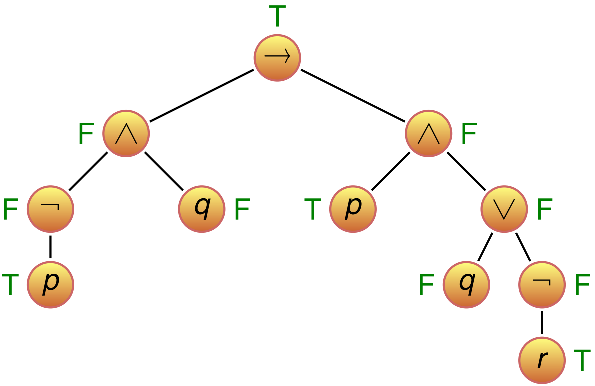

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F			
F	T			
T	F			
T	T			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T		
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F	F	T	T	
F	T	F		
T	F	T		
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T		
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T	T	
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	
T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \land (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \lor \neg p$
F	T	T
T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

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p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F				
F	T				
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T			
F	T				
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T	T			
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T	T			
T	F	F			
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T	T			
T	F	F			
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T			
T	F	F			
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

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p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T	T		
T	F	F			
T	T	F			

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F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

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F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	F			

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p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	F	T		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	
F	T	T	T		
T	F	F	F		
T	T	F	T		

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F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

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F	F	T	T	F	
F	T	T	T	F	
T	F	F	F		
T	T	F	T		

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p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T		

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F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

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F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T	F	

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F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

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F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \land (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \lor \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \land \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	
T	T	F	T	F	

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p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

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p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

contradiction: always F

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

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Formulas ϕ and ψ are **logically equivalent**, denoted

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p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \to q$$

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

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if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \rightarrow q$$

More examples

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$

$$p \vee q \equiv q \vee p$$

$$p \rightarrow \neg q \equiv q \rightarrow \neg p$$

Important Equivalences

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$$\neg\neg\alpha \equiv \alpha$$

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$$\alpha \wedge \alpha \equiv \alpha$$

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$$\neg\neg\alpha \equiv \alpha$$

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$$\alpha \vee \alpha \equiv \alpha$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\
\alpha \wedge \beta &\equiv \beta \wedge \alpha
\end{aligned}
$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\[1ex]
\alpha \wedge \beta &\equiv \beta \wedge \alpha \\
\alpha \vee \beta &\equiv \beta \vee \alpha
\end{aligned}
$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\[1ex]
\alpha \wedge \beta &\equiv \beta \wedge \alpha \\
\alpha \vee \beta &\equiv \beta \vee \alpha \\[1ex]
\alpha \rightarrow \beta &\equiv \neg\alpha \vee \beta
\end{aligned}
$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\[1ex]
\alpha \wedge \beta &\equiv \beta \wedge \alpha \\
\alpha \vee \beta &\equiv \beta \vee \alpha \\[1ex]
\alpha \rightarrow \beta &\equiv \neg\alpha \vee \beta \\
\alpha \rightarrow \beta &\equiv \neg\beta \rightarrow \neg\alpha
\end{aligned}
$$

Important Equivalences

$$\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \land \alpha &\equiv \alpha \\
\alpha \lor \alpha &\equiv \alpha \\[1ex]
\alpha \land \beta &\equiv \beta \land \alpha \\
\alpha \lor \beta &\equiv \beta \lor \alpha \\[1ex]
\alpha \to \beta &\equiv \neg\alpha \lor \beta \\
\alpha \to \beta &\equiv \neg\beta \to \neg\alpha
\end{aligned}$$

De Morgan laws

$$\neg(\alpha \lor \beta) \equiv \neg\alpha \land \neg\beta$$

Important Equivalences

$$\begin{aligned}
\neg\neg\alpha &\equiv \alpha \\[1ex]
\alpha \wedge \alpha &\equiv \alpha \\
\alpha \vee \alpha &\equiv \alpha \\[1ex]
\alpha \wedge \beta &\equiv \beta \wedge \alpha \\
\alpha \vee \beta &\equiv \beta \vee \alpha \\[1ex]
\alpha \rightarrow \beta &\equiv \neg\alpha \vee \beta \\
\alpha \rightarrow \beta &\equiv \neg\beta \rightarrow \neg\alpha
\end{aligned}$$

De Morgan laws

$$\begin{aligned}
\neg(\alpha \vee \beta) &\equiv \neg\alpha \wedge \neg\beta \\
\neg(\alpha \wedge \beta) &\equiv \neg\alpha \vee \neg\beta
\end{aligned}$$

Logical Reasoning

From $\alpha_1, \ldots, \alpha_n$ follows β.

This is denoted as:

$$\alpha_1, \ldots, \alpha_n \;\therefore\; \beta$$

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$$p \rightarrow (q \rightarrow r),\ p,\ \neg r \;\therefore$$

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Logical Reasoning

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$$p \wedge \neg q \;\therefore\; \neg q$$

$$p \rightarrow (q \rightarrow r),\; p,\; \neg r \;\therefore\; \neg q$$

Two Important Variants

semantic	$\alpha_1, \ldots, \alpha_n \models \beta$
syntactic	$\alpha_1, \ldots, \alpha_n \vdash \beta$

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \to q$?

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

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Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \rightarrow q$ is T.

Semantic Entailment

Semantic Entailment / Consequence

$$\alpha_1, \ldots, \alpha_n \models \beta$$

means

Whenever $\alpha_1, \ldots, \alpha_n$ are all true, β is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \rightarrow q$ is T. Hence: $q \models p \rightarrow q$.

Examples Semantic Entailment

Do we have $p \to q, \neg q \models \neg p$?

Examples Semantic Entailment

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

Examples Semantic Entailment

Do we have $p \to q, \neg q \models \neg p$?

p	q	$p \to q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

Examples Semantic Entailment

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \rightarrow q$ and $\neg q$ are T

Examples Semantic Entailment

Do we have $p \to q, \neg q \models \neg p$?

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In this line(s) $\neg p$ is true.

Hence $p \to q, \neg q \models \neg p$ holds.

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When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

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When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

Conclusion: $p \vee q \not\models p \rightarrow q$.

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A formula ϕ is a tautology if it holds without premises:

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$$\alpha \equiv \beta \quad \Longleftrightarrow \quad \alpha \models \beta \ \text{ and } \ \beta \models \alpha$$

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Other Important Facts

$$\alpha \models \beta \quad \Longleftrightarrow \quad \models \alpha \rightarrow \beta$$

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$$\alpha \models \beta \quad \iff \quad \models \alpha \to \beta$$

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For example: $\models p \vee \neg p$, but not $\models p$ and not $\models \neg p$.