Logic and Modelling

— Propositional Logic —

Jörg Endrullis

VU University Amsterdam

Propositional Logic: Introduction

Propositions are “declarative sentences”.

Examples

It rains.

$8 > 5$.

$6 < 2$.

X is positive.

The earth is flat.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella,
  then she is wet.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus,

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p &: \quad \text{It rains.} \\ q &: \quad \text{Janneke has her umbrella.} \\ r &: \quad \text{Janneke is wet.} \end{aligned}$$

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

p : It rains.

q : Janneke has her umbrella.

r : Janneke is wet.

Then the sentences above becomes:

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p:&\quad \text{It rains.} \\ q:&\quad \text{Janneke has her umbrella.} \\ r:&\quad \text{Janneke is wet.} \end{aligned}$$

Then the sentences above becomes:

$$(p \wedge \neg q) \rightarrow r$$

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p: &\quad \text{It rains.} \\ q: &\quad \text{Janneke has her umbrella.} \\ r: &\quad \text{Janneke is wet.} \end{aligned}$$

Then the sentences above becomes:

$$(p \wedge \neg q) \to r, \quad \neg r$$

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p:&\quad \text{It rains.}\\ q:&\quad \text{Janneke has her umbrella.}\\ r:&\quad \text{Janneke is wet.} \end{aligned}$$

Then the sentences above becomes:

$$(p \wedge \neg q) \rightarrow r, \quad \neg r, \quad p$$

Propositional Logic: Introduction

- **If** it rains **and** Janneke does **not** have her umbrella, **then** she is wet.
- Janneke is **not** wet.
- It rains.

Thus, Janneke has her umbrella.

We can translate this to propositional logic as follows:

$$\begin{aligned} p &: \quad \text{It rains.} \\ q &: \quad \text{Janneke has her umbrella.} \\ r &: \quad \text{Janneke is wet.} \end{aligned}$$

Then the sentences above becomes:

$$(p \wedge \neg q) \rightarrow r, \quad \neg r, \quad p \quad \models \quad q$$

Propositional Logic: Syntax

Syntax of Propositional Logic

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p, \ q, \ r, \ \ldots \quad p_0, \ p_1, \ \ldots$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p \qquad\qquad p \wedge q$$

$$\neg p \qquad\qquad p \vee \neg q$$

$$(\neg p \wedge q) \rightarrow r$$

Propositional Logic: Syntax

Syntax of Propositional Logic

- propositional variables

$$p,\ q,\ r,\ \ldots \quad p_0,\ p_1,\ \ldots$$

- propositional connectives

$$\neg,\ \wedge,\ \vee,\ \rightarrow$$

- propositional formulas

$$p$$

$$\neg p$$

$$(\neg p \wedge q) \rightarrow r$$

$$p \wedge q$$

$$p \vee \neg q$$

We use Greek letters for formulas:

ϕ	phi	ψ	psi	ξ	xi
α	alpha	β	beta	$\ldots$	

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$p \wedge q \rightarrow r \quad =$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad =$$

Bracket Convention

Priority Rules

$$\begin{array}{rl} \neg & \text{binds the strongest} \\ \rightarrow & \text{binds the weakest} \\ \wedge, \vee & \text{binding-strength is in-between} \end{array}$$

How do we read the following?

$$\begin{aligned} p \wedge q \rightarrow r &= (p \wedge q) \rightarrow r \\ \neg p \wedge r &= (\neg p) \wedge r \end{aligned}$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\to$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$
\begin{aligned}
p \wedge q \to r &= (p \wedge q) \to r \\
\neg p \wedge r &= (\neg p) \wedge r \\
p \wedge q \vee r &=
\end{aligned}
$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

$p \wedge q \vee r \quad = \quad$ **unclear! needs brackets!**

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

$p \wedge q \vee r \quad =$ **unclear! needs brackets!**

$$p \wedge q \wedge r \quad =$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

$p \wedge q \vee r \quad = \quad$ **unclear! needs brackets!**

$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$
\begin{aligned}
p \wedge q \rightarrow r &= (p \wedge q) \rightarrow r \\
\neg p \wedge r &= (\neg p) \wedge r \\
p \wedge q \vee r &= \textbf{unclear! needs brackets!} \\
p \wedge q \wedge r &= (p \wedge q) \wedge r = p \wedge (q \wedge r)
\end{aligned}
$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

$$p \wedge q \vee r \quad = \quad \textbf{unclear! needs brackets!}$$

$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)}$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$
\begin{aligned}
p \wedge q \rightarrow r &= (p \wedge q) \rightarrow r \\
\neg p \wedge r &= (\neg p) \wedge r \\
p \wedge q \vee r &= \textbf{unclear! needs brackets!} \\
p \wedge q \wedge r &= (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)} \\
p \rightarrow q \rightarrow r &=
\end{aligned}
$$

Bracket Convention

Priority Rules

$\neg$	binds the strongest
$\rightarrow$	binds the weakest
$\wedge, \vee$	binding-strength is in-between

How do we read the following?

$$p \wedge q \rightarrow r \quad = \quad (p \wedge q) \rightarrow r$$

$$\neg p \wedge r \quad = \quad (\neg p) \wedge r$$

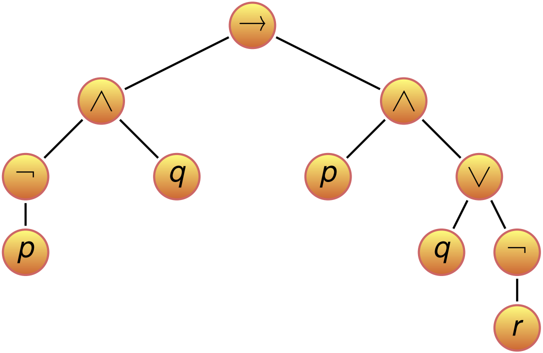
$p \wedge q \vee r \quad = \quad$ **unclear! needs brackets!**

$$p \wedge q \wedge r \quad = \quad (p \wedge q) \wedge r \quad = \quad p \wedge (q \wedge r) \quad \text{(associativity)}$$

$p \rightarrow q \rightarrow r \quad = \quad$ **unclear! needs brackets! (not associative)**

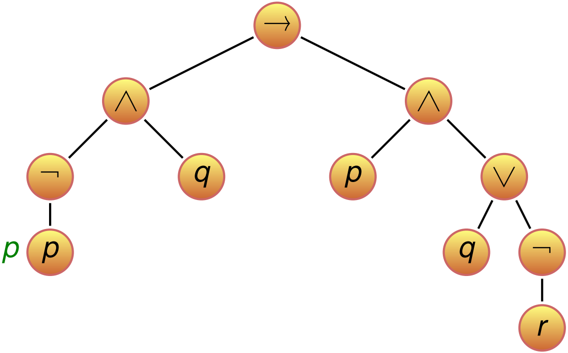
Parse Trees

What is the formula corresponding to the following parse tree?



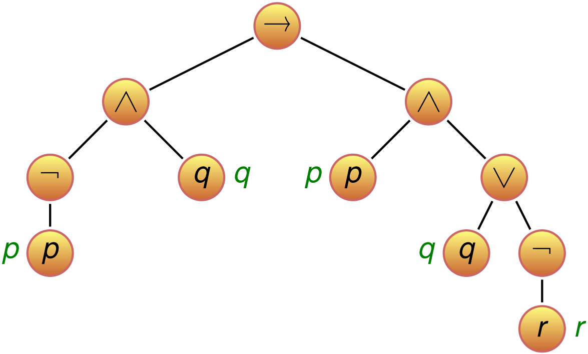
Parse Trees

What is the formula corresponding to the following parse tree?



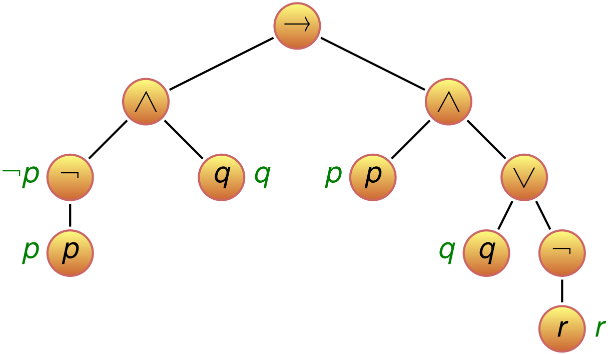
Parse Trees

What is the formula corresponding to the following parse tree?



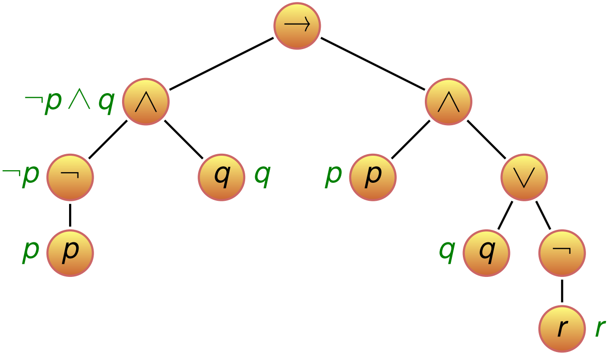
Parse Trees

What is the formula corresponding to the following parse tree?



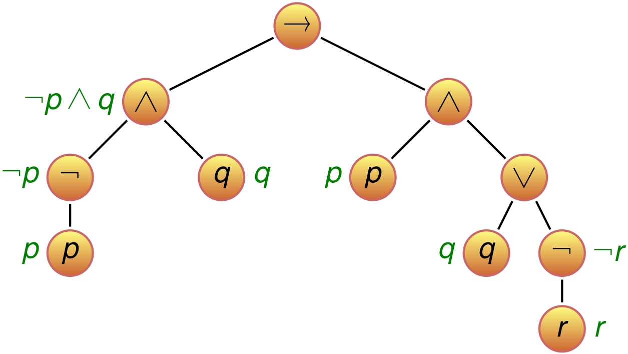
Parse Trees

What is the formula corresponding to the following parse tree?



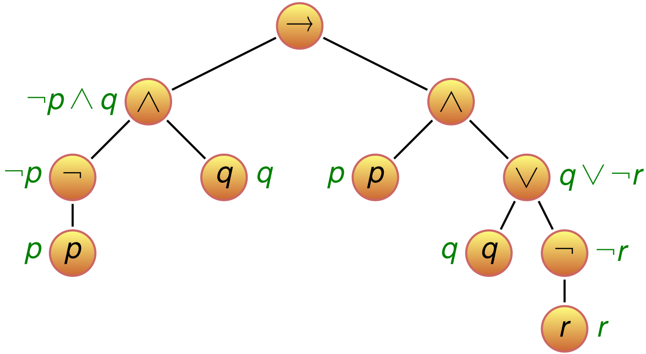
Parse Trees

What is the formula corresponding to the following parse tree?



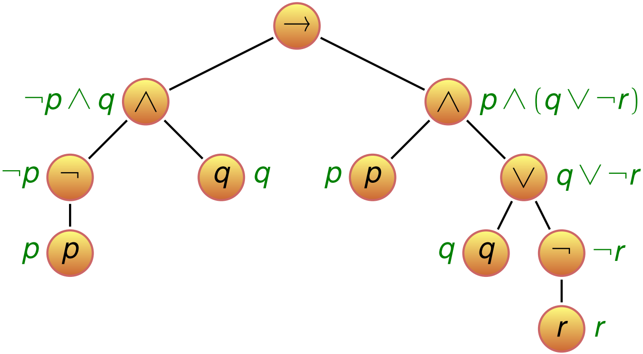
Parse Trees

What is the formula corresponding to the following parse tree?



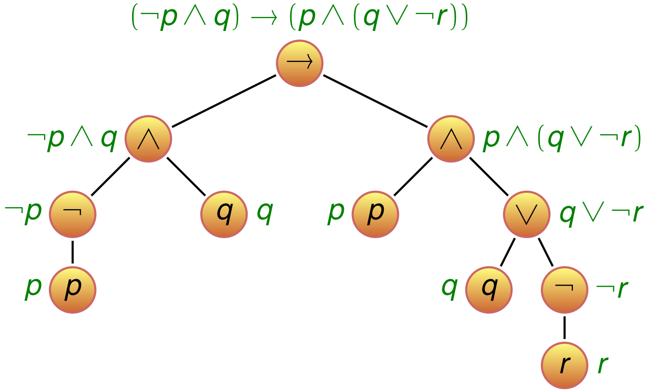
Parse Trees

What is the formula corresponding to the following parse tree?



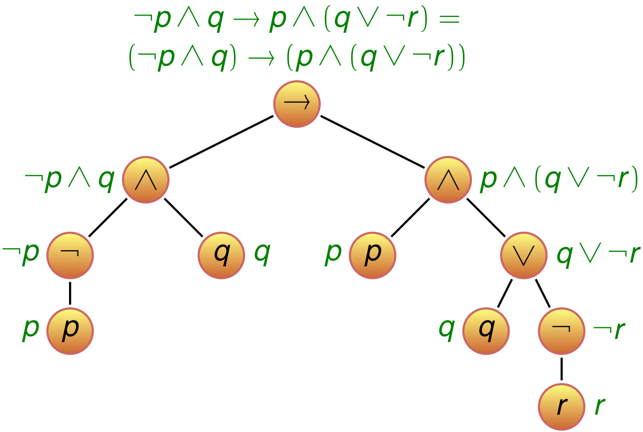
Parse Trees

What is the formula corresponding to the following parse tree?



Parse Trees

What is the formula corresponding to the following parse tree?



Properties of Formulas

Contingency

Tautology

Contradiction

Properties of Formulas

Contingency = sometimes true and sometimes false

Tautology

Contradiction

Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology

Contradiction

Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

Contradiction

Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \rightarrow p$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

Contradiction

Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \rightarrow p$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

Contradiction = always false

Properties of Formulas

Contingency = sometimes true and sometimes false

$$p \wedge q \wedge \neg r$$

Tautology = always true

$$p \vee \neg p$$

$$p \rightarrow p$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

Contradiction = always false

$$p \wedge \neg p$$

$$((p \rightarrow q) \rightarrow p) \wedge \neg p$$

Properties of Formulas

Equivalent formulas

Properties of Formulas

Equivalent formulas = true at the same time

Properties of Formulas

Equivalent formulas = true at the same time

$$\neg p \vee q \;\equiv\; p \rightarrow q$$

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

Truth Tables

ϕ	ψ	$\phi \land \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \lor \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \to \psi$
F	F	T
F	T	T
T	F	F
T	T	T

$\bot$
F

Truth Tables

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

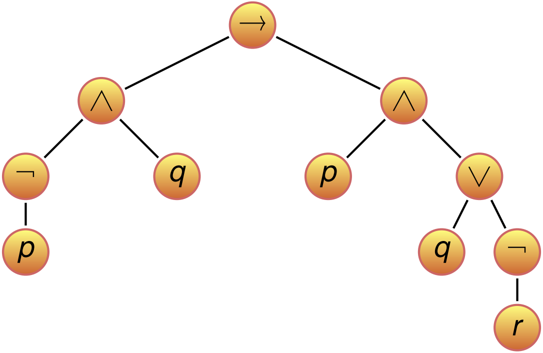
$\bot$
F

$\top$
T

Parse Trees

Bottom-up evaluation of truth values in a parse tree:

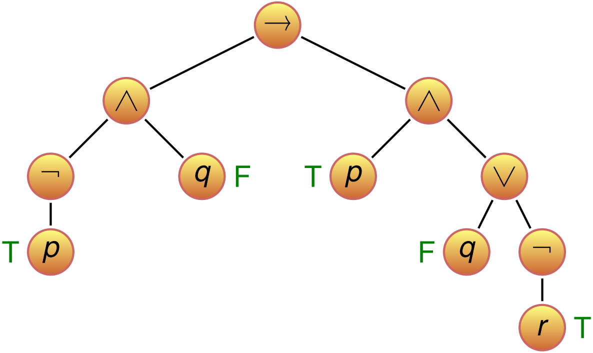
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

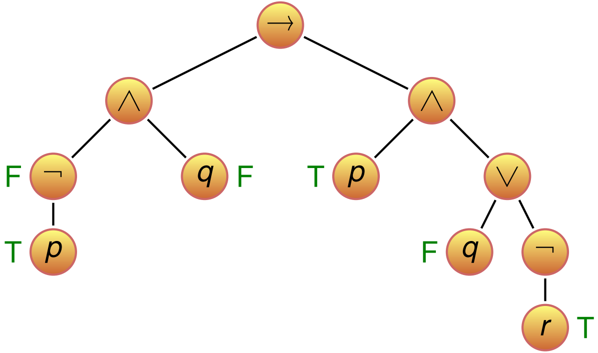
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

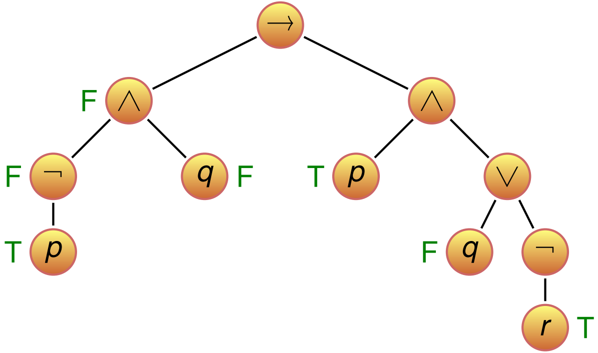
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

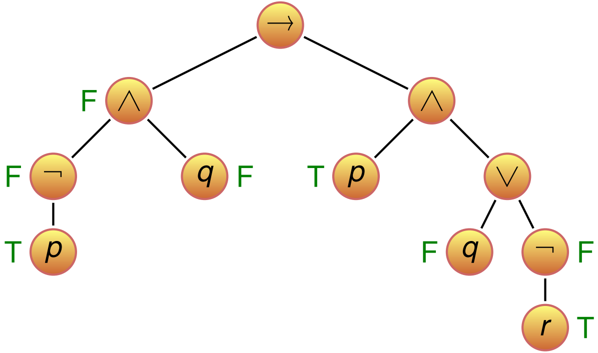
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

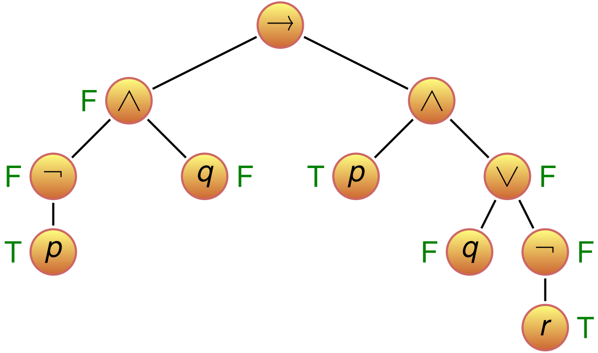
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

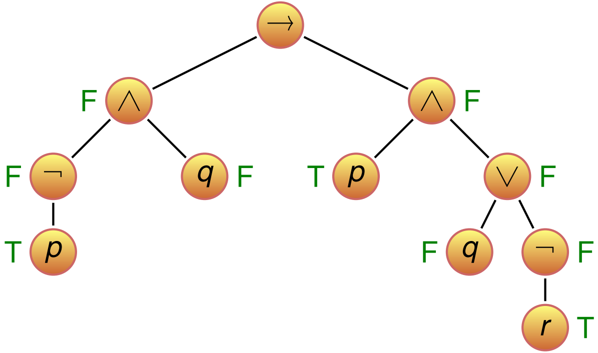
- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

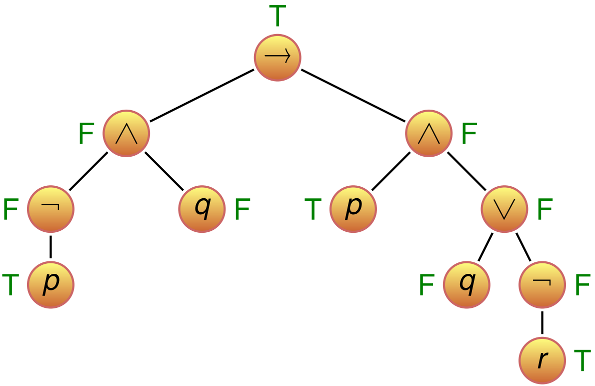
- $p = \mathsf{T}$
- $q = \mathsf{F}$
- $r = \mathsf{T}$



Parse Trees

Bottom-up evaluation of truth values in a parse tree:

- $p = \mathrm{T}$
- $q = \mathrm{F}$
- $r = \mathrm{T}$



Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F			
F	T			
T	F			
T	T			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T		
F	T			
T	F			
T	T			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T		
F	T	F		
T	F			
T	T			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T		
F	T	F		
T	F	T		
T	T			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \land (p \rightarrow \neg q)$
F	F	T		
F	T	F		
T	F	T		
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	
F	T	F		
T	F	T		
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \land (p \to \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T		
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T	T	
T	T	F		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	
F	T	F	T	
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	
T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F				
F	T				
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T				
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T	T			
T	F				
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T			
F	T	T			
T	F	F			
T	T				

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \land (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \lor \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \land \neg(p \to q)$
F	F	T			
F	T	T			
T	F	F			
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T			
T	F	F			
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T		
F	T	T	T		
T	F	F			
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	F			

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	F	T		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	
F	T	T	T		
T	F	F	F		
T	T	F	T		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	
F	T	T	T	F	
T	F	F	F		
T	T	F	T		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T		

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	F
F	T	T	T	F	
T	F	F	F	T	
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \wedge \neg(p \rightarrow q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \wedge \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

Truth Tables and Properties of Formulas

p	q	$\neg q$	$p \to \neg q$	$p \land (p \to \neg q)$
F	F	T	T	F
F	T	F	T	F
T	F	T	T	T
T	T	F	F	F

contingency: sometimes F, sometimes T

p	$\neg p$	$p \lor \neg p$
F	T	T
T	F	T

tautology: always T

p	q	$\neg p$	$p \to q$	$\neg(p \to q)$	$\neg p \land \neg(p \to q)$
F	F	T	T	F	F
F	T	T	T	F	F
T	F	F	F	T	F
T	T	F	T	F	F

contradiction: always F

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi ,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \to q$$

Logic Equivalence

Formulas ϕ and ψ are **logically equivalent**, denoted

$$\phi \equiv \psi\,,$$

if ϕ and ψ have the same truth table.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$$\neg p \vee q \equiv p \rightarrow q$$

More examples

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$

$$p \vee q \equiv q \vee p$$

$$p \rightarrow \neg q \equiv q \rightarrow \neg p$$

Important Equivalences

Important Equivalences

$$\neg\neg\phi \equiv \phi$$

Important Equivalences

$$\neg\neg\phi \equiv \phi$$

$$\phi \land \phi \equiv \phi$$

Important Equivalences

$$\neg\neg\phi \equiv \phi$$

$$\phi \wedge \phi \equiv \phi$$

$$\phi \vee \phi \equiv \phi$$

Important Equivalences

$$\neg\neg\phi \equiv \phi$$

$$\phi \wedge \phi \equiv \phi$$

$$\phi \vee \phi \equiv \phi$$

$$\phi \wedge \psi \equiv \psi \wedge \phi$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\[1ex]
\phi \wedge \phi &\equiv \phi \\
\phi \vee \phi &\equiv \phi \\[1ex]
\phi \wedge \psi &\equiv \psi \wedge \phi \\
\phi \vee \psi &\equiv \psi \vee \phi
\end{aligned}
$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\[1ex]
\phi \land \phi &\equiv \phi \\
\phi \lor \phi &\equiv \phi \\[1ex]
\phi \land \psi &\equiv \psi \land \phi \\
\phi \lor \psi &\equiv \psi \lor \phi \\[1ex]
\phi \rightarrow \psi &\equiv \neg\phi \lor \psi
\end{aligned}
$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\[1ex]
\phi \wedge \phi &\equiv \phi \\
\phi \vee \phi &\equiv \phi \\[1ex]
\phi \wedge \psi &\equiv \psi \wedge \phi \\
\phi \vee \psi &\equiv \psi \vee \phi \\[1ex]
\phi \rightarrow \psi &\equiv \neg\phi \vee \psi \\
\phi \rightarrow \psi &\equiv \neg\psi \rightarrow \neg\phi
\end{aligned}
$$

Important Equivalences

$$\neg\neg\phi \equiv \phi$$

$$\phi \wedge \phi \equiv \phi$$

$$\phi \vee \phi \equiv \phi$$

$$\phi \wedge \psi \equiv \psi \wedge \phi$$

$$\phi \vee \psi \equiv \psi \vee \phi$$

$$\phi \to \psi \equiv \neg\phi \vee \psi$$

$$\phi \to \psi \equiv \neg\psi \to \neg\phi$$

De Morgan laws

$$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$

Important Equivalences

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\[1ex]
\phi \wedge \phi &\equiv \phi \\
\phi \vee \phi &\equiv \phi \\[1ex]
\phi \wedge \psi &\equiv \psi \wedge \phi \\
\phi \vee \psi &\equiv \psi \vee \phi \\[1ex]
\phi \rightarrow \psi &\equiv \neg\phi \vee \psi \\
\phi \rightarrow \psi &\equiv \neg\psi \rightarrow \neg\phi
\end{aligned}
$$

De Morgan laws

$$
\begin{aligned}
\neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi \\
\neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi
\end{aligned}
$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ **follows** ψ**.**

This is denoted as:

$$\phi_1, \ldots, \phi_n \ \therefore\ \psi$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ follows ψ.

This is denoted as:

$$\phi_1, \ldots, \phi_n \;\therefore\; \psi$$

$$p \wedge \neg q \;\therefore\; \neg q$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ follows ψ.

This is denoted as:

$$\phi_1, \ldots, \phi_n \;\therefore\; \psi$$

$$p \wedge \neg q \;\therefore\; \neg q$$

$$p \rightarrow (q \rightarrow r),\; p,\; \neg r \;\therefore$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ follows ψ.

This is denoted as:

$$\phi_1, \ldots, \phi_n \;\therefore\; \psi$$

$$p \wedge \neg q \;\therefore\; \neg q$$

$$p \rightarrow (q \rightarrow r),\; p,\; \neg r \;\therefore\; \neg q$$

Logical Reasoning

From $\phi_1, \ldots, \phi_n$ follows ψ.

This is denoted as:

$$\phi_1, \ldots, \phi_n \;\therefore\; \psi$$

$$p \wedge \neg q \;\therefore\; \neg q$$

$$p \rightarrow (q \rightarrow r),\; p,\; \neg r \;\therefore\; \neg q$$

Two Important Variants

semantic	$\phi_1, \ldots, \phi_n \models \psi$
syntactic	$\phi_1, \ldots, \phi_n \vdash \psi$

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \to q$?

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \to q$?

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \to q$?

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \rightarrow q$ is T.

Semantic Implication

Semantic Implication / Consequence

$$\phi_1, \ldots, \phi_n \models \psi$$

means

Whenever $\phi_1, \ldots, \phi_n$ are all true, ψ is also true.

Do we have $q \models p \rightarrow q$?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Whenever q is T also $p \rightarrow q$ is T. Hence: $q \models p \rightarrow q$.

Examples Semantic Implication

Do we have $p \rightarrow q, \neg q \models \neg p$?

Examples Semantic Implication

Do we have $p \to q, \neg q \models \neg p$?

p	q	$p \to q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

Examples Semantic Implication

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

Examples Semantic Implication

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \rightarrow q$ and $\neg q$ are T

Examples Semantic Implication

Do we have $p \rightarrow q, \neg q \models \neg p$?

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \rightarrow q$ and $\neg q$ are T

In this line(s) $\neg p$ is true.

Examples Semantic Implication

Do we have $p \to q, \neg q \models \neg p$?

p	q	$p \to q$	$\neg q$	$\neg p$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	T

At which line(s) do we need to look?

- where both $p \to q$ and $\neg q$ are T

In this line(s) $\neg p$ is true.

Hence $p \to q, \neg q \models \neg p$ holds.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $\quad p \to q, \neg q \models \neg p \quad$?

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $p \to q, \neg q \models \neg p$?

Assume that $p \to q$ and $\neg q$ are T.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $p \to q, \neg q \models \neg p$?

Assume that $p \to q$ and $\neg q$ are T.

Then q is F.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $p \rightarrow q, \neg q \models \neg p$?

Assume that $p \rightarrow q$ and $\neg q$ are T.

Then q is F.

Then p must be F since otherwise $p \rightarrow q$ was F.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $\quad p \rightarrow q, \neg q \models \neg p \quad$?

Assume that $p \rightarrow q$ and $\neg q$ are T.

Then q is F.

Then p must be F since otherwise $p \rightarrow q$ was F.

Thus $\neg p$ is T.

Examples Semantic Implication

To show that $\phi_1, \ldots, \phi_n \models \psi$ we need to show

- ψ is true whenever $\phi_1, \ldots, \phi_n$ are true.

This can also be achieved by logical reasoning:

- assume that $\phi_1, \ldots, \phi_n$ are true, and
- show that ψ must be true as well.

Do we have $p \to q, \neg q \models \neg p$?

Assume that $p \to q$ and $\neg q$ are T.

Then q is F.

Then p must be F since otherwise $p \to q$ was F.

Thus $\neg p$ is T.

Hence $p \to q, \neg q \models \neg p$ holds.

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$?

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $p \vee q \models p \rightarrow q$?

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $\quad p \vee q \models p \rightarrow q \quad$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $p \vee q \models p \rightarrow q$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $p \vee q \models p \rightarrow q$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

Disproving Semantic Implication

How to disprove $\phi_1, \ldots, \phi_n \models \psi$? That is, $\phi_1, \ldots, \phi_n \not\models \psi$?

Find a valuation (assignment of truth values to variables) that

- makes $\phi_1, \ldots, \phi_n$ true, and
- ψ false.

Do we have $\quad p \vee q \models p \rightarrow q \quad$?

p	q	$p \vee q$	$p \rightarrow q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

When p is T and q is F, then $p \vee q$ is T and $p \rightarrow q$ is F.

Conclusion: $p \vee q \not\models p \rightarrow q$.

Examples Semantic Implication

Which of the following semantic implications hold?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models\ p \rightarrow q$?
- $q \models\ p \rightarrow q$?
- $p \models\ p \rightarrow q$?
- $p \rightarrow q \models\ p$?
- $p \rightarrow q \models\ q$?
- $p \vee q \models\ q \vee p$?
- $p \vee q \models\ p$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \rightarrow q$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \rightarrow q$?
- $p \rightarrow (q \rightarrow r) \models q \rightarrow (p \rightarrow r)$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \to q$?
- $q \models p \to q$?
- $p \models p \to q$?
- $p \to q \models p$?
- $p \to q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \to q$?
- $p \to (q \to r) \models q \to (p \to r)$?
- $p \to (q \to r) \models (p \to q) \to r$?

Examples Semantic Implication

Which of the following semantic implications hold?

- $p \vee q,\ q \models p \rightarrow q$?
- $q \models p \rightarrow q$?
- $p \models p \rightarrow q$?
- $p \rightarrow q \models p$?
- $p \rightarrow q \models q$?
- $p \vee q \models q \vee p$?
- $p \vee q \models p$?
- $p \wedge q \models p \rightarrow q$?
- $p \rightarrow (q \rightarrow r) \models q \rightarrow (p \rightarrow r)$?
- $p \rightarrow (q \rightarrow r) \models (p \rightarrow q) \rightarrow r$?
- $(p \rightarrow q) \rightarrow r \models p \rightarrow (q \rightarrow r)$?

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \Longleftrightarrow \quad \phi \text{ is a tautology}$$

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \iff \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \Longleftrightarrow \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad \Longleftrightarrow \quad \phi \models \psi \text{ and } \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \iff \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad \iff \quad \phi \models \psi \text{ and } \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Note that $\equiv$ is an equivalence relation:

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad\iff\quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad\iff\quad \phi \models \psi \ \text{ and } \ \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Note that $\equiv$ is an equivalence relation:

- reflexive,

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \Longleftrightarrow \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad \Longleftrightarrow \quad \phi \models \psi \;\text{ and }\; \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Note that $\equiv$ is an equivalence relation:

- reflexive,
- symmetric,

Tautologies and Semantic Equivalence

Tautology

A formula ϕ is a tautology if it holds without premises:

$$\models \phi \quad \iff \quad \phi \text{ is a tautology}$$

$$\models p \vee \neg p$$

Semantic Equivalence

$$\phi \equiv \psi \quad \iff \quad \phi \models \psi \text{ and } \psi \models \phi$$

(In other words: ϕ and ψ have the same truth table.)

Note that $\equiv$ is an equivalence relation:

- reflexive,
- symmetric,
- transitive.

Other Important Facts

$$\phi \models \psi \quad \iff \quad \models \phi \rightarrow \psi$$

Other Important Facts

$$\phi \models \psi \quad \iff \quad \models \phi \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \iff \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \rightarrow \psi$$

Other Important Facts

$$\phi \models \psi \quad \Longleftrightarrow \quad \models \phi \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))$$

Other Important Facts

$$\phi \models \psi \quad \Longleftrightarrow \quad \models \phi \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))$$

$$\models \phi \wedge \psi \quad \Longleftrightarrow \quad \models \phi \text{ and } \models \psi$$

Other Important Facts

$$\phi \models \psi \iff \models \phi \to \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \iff \phi_1, \ldots, \phi_{n-1} \models \phi_n \to \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \iff \models \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$$

$$\models \phi \wedge \psi \iff \models \phi \text{ and } \models \psi$$

$$\models \phi \text{ or } \models \psi \implies \models \phi \vee \psi$$

Other Important Facts

$$\phi \models \psi \quad \Longleftrightarrow \quad \models \phi \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$$

$$\models \phi \wedge \psi \quad \Longleftrightarrow \quad \models \phi \text{ and } \models \psi$$

$$\models \phi \text{ or } \models \psi \quad \Longrightarrow \quad \models \phi \vee \psi$$

$$\models \phi \vee \psi \quad \not\Longrightarrow \quad \models \phi \text{ or } \models \psi$$

Other Important Facts

$$\phi \models \psi \quad \Longleftrightarrow \quad \models \phi \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \rightarrow \psi$$

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$$

$$\models \phi \wedge \psi \quad \Longleftrightarrow \quad \models \phi \text{ and } \models \psi$$

$$\models \phi \text{ or } \models \psi \quad \Longrightarrow \quad \models \phi \vee \psi$$

$$\models \phi \vee \psi \quad \not\Longrightarrow \quad \models \phi \text{ or } \models \psi$$

For example: $\models p \vee \neg p$, but not $\models p$ and not $\models \neg p$.