Calculus M211

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Indiana University Bloomington

2013

Average Value of a Function

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- $s(t)$ the position at time t
- $v(t)$ the speed at time t

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The Mean Value Theorem tell us that:

There exists c in (a, b) such that $v(c) = v_{\text{avg}} = \frac{1}{b-a}\int_a^b v(t)dt$.

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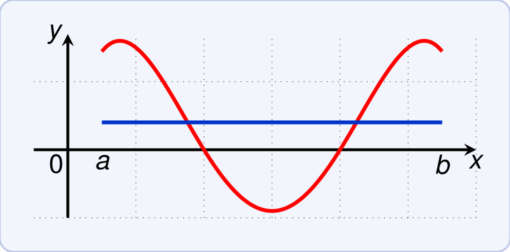
How to compute the average value of a function?



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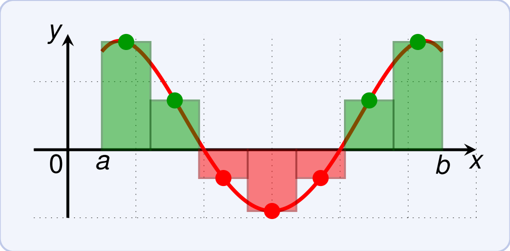

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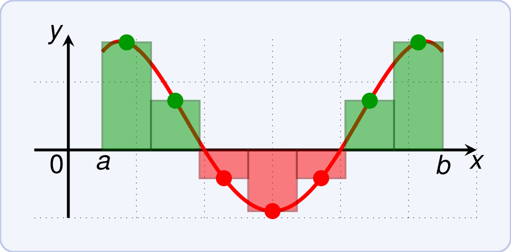

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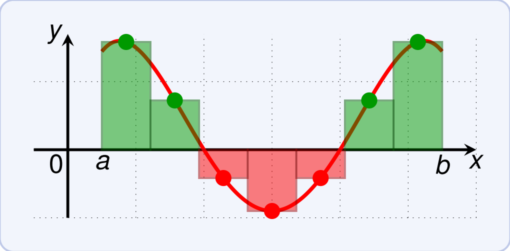

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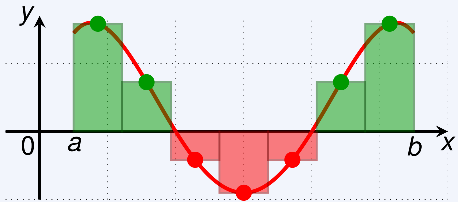

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$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n\,\Delta x}\sum_{i=1}^{n} f(x_i)\Delta x$$

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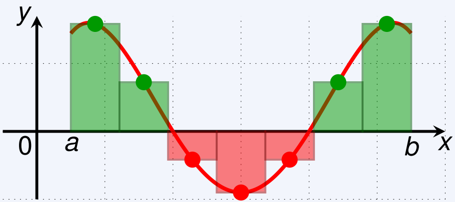

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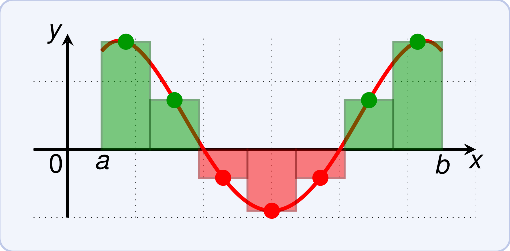

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The sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is called **Riemann sum.**

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As a consequence, we have

The average value f_{avg} of a function f on an interval $[a, b]$ is:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Thus the average value of the function is the integral over the interval divided by the width of the interval.

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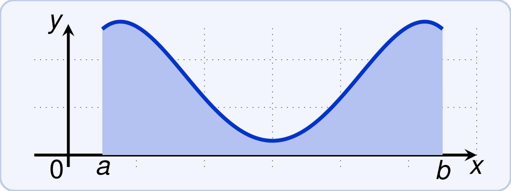

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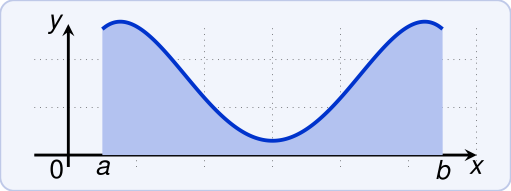

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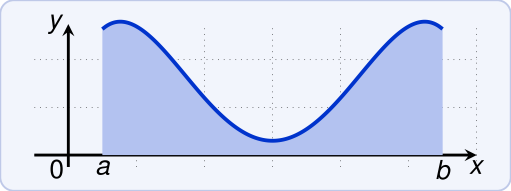

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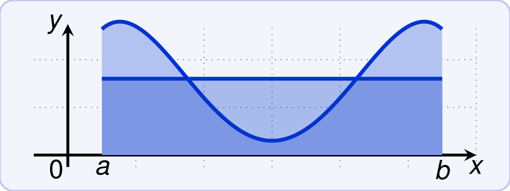

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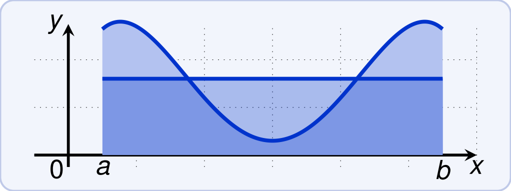

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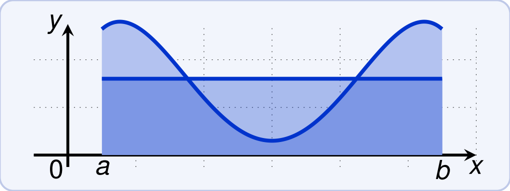

- Think of the area below the function as water.
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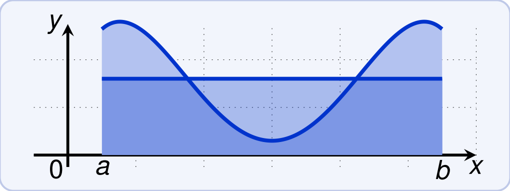

- Think of the area below the function as water.
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- When the waves calm, the water settles in the shape of a rectangle with area A and width $b-a$; thus height $\frac{A}{b-a}$

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that is,

$$f(c)(b-a) \;=\; \int_a^b f(x)\,dx$$

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Thus there are two numbers $c = -1$ and $c = 1$ that work!