

Calculus M211

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The Substitution Rule for Indefinite Integrals

How to find antiderivatives of formulas like

$$\int 2x\sqrt{1+x^2}dx \quad ?$$

Recall the chain rule:

$$(f(g(x)))' = f'(g(x))g'(x)$$

Let us try to write the integral in the form:

$$\int f'(g(x))g'(x)dx$$

Then

$$f'(x) = \sqrt{x} \quad g(x) = 1 + x^2 \quad g'(x) = 2x$$

Moreover the antiderivative of $f'(x)$ is $f(x) = \frac{2}{3}x^{\frac{3}{2}}$. Thus

$$\int 2x\sqrt{1+x^2}dx = f(g(x)) + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

To remember this rule: note that if $u = g(x)$, then

$$du = g'(x)dx$$

(here we think of dx and du as differentials)

In other words:

$$dx = \frac{du}{g'(x)}$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

$$\int 2x\sqrt{1+x^2} dx$$

We choose $u = 1 + x^2$. Then $u' = 2x$, and hence

$$\begin{aligned}\int 2x\sqrt{1+x^2} dx &= \int 2x\sqrt{u} \frac{du}{2x} = \int \sqrt{u} du \\ &= \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (1+x^2)^{\frac{3}{2}} + C\end{aligned}$$

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

$$\int x^3 \cos(x^4 + 2) dx$$

We choose $u = x^4 + 2$. Then $u' = 4x^3$, and hence

$$\begin{aligned} \int x^3 \cos(x^4 + 2) dx &= \int x^3 \cos(u) \frac{du}{4x^3} = \frac{1}{4} \int \cos(u) du \\ &= \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4 + 2) + C \end{aligned}$$

Finding the right u is a guessing game. Often multiple choices.

The Substitution Rule for Indefinite Integrals

$$\int \sqrt{2x+1} dx$$

We choose $u = 2x + 1$. Then $u' = 2$, and hence

$$\begin{aligned}\int \sqrt{2x+1} dx &= \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int \sqrt{u} du \\ &= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C\end{aligned}$$

We could have chosen another u . For example:

We choose $u = \sqrt{2x+1}$. Then $u' = \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{1}{u}$. Then

$$\begin{aligned}\int \sqrt{2x+1} dx &= \int u \frac{du}{1/u} = \int u^2 du \\ &= \frac{1}{3} u^3 + C = \frac{1}{3} (\sqrt{2x+1})^3 + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C\end{aligned}$$

The Substitution Rule for Indefinite Integrals

$$\int \frac{x}{\sqrt{1-4x^2}} dx$$

We choose $u = 1 - 4x^2$. Then $u' = -8x$, and hence

$$\begin{aligned} \int \frac{x}{\sqrt{1-4x^2}} dx &= \int \frac{x}{\sqrt{u}} \frac{du}{-8x} = -\frac{1}{8} \int \frac{1}{\sqrt{u}} du \\ &= -\frac{1}{8} \cdot 2\sqrt{u} + C = -\frac{1}{4} \sqrt{1-4x^2} + C \end{aligned}$$

$$\int e^{5x} dx$$

We choose $u = 5x$. Then $u' = 5$, and hence

$$\begin{aligned} \int e^{5x} dx &= \int e^u \frac{du}{5} = \frac{1}{5} \int e^u du \\ &= \frac{1}{5} e^u + C = \frac{1}{5} e^{5x} + C \end{aligned}$$

The Substitution Rule for Indefinite Integrals

A slightly more interesting example:

$$\int x^5 \sqrt{1+x^2} dx$$

We choose $u = 1 + x^2$. Then $u' = 2x$, and hence

$$\int x^5 \sqrt{1+x^2} dx = \int x^5 \sqrt{u} \frac{du}{2x} = \frac{1}{2} \int x^4 \sqrt{u} du$$

What now? Note that $x^2 = u - 1$ and $x^4 = (x^2)^2$

$$\begin{aligned} \int x^5 \sqrt{1+x^2} dx &= \frac{1}{2} \int x^4 \sqrt{u} du = \frac{1}{2} \int (u-1)^2 \sqrt{u} du \\ &= \frac{1}{2} \int (u^2 - 2u + 1) \sqrt{u} du = \frac{1}{2} \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) du \\ &= \frac{1}{2} \left(\frac{2}{7} u^{\frac{7}{2}} - 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right) + C \\ &= \frac{1}{7} (1+x^2)^{\frac{7}{2}} - \frac{2}{5} (1+x^2)^{\frac{5}{2}} + \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C \end{aligned}$$

The Substitution Rule for Indefinite Integrals

$$\int \tan x \, dx$$

First, we note that:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

We choose $u = \cos x$. Then $u' = -\sin x$, and hence

$$\begin{aligned} \int \frac{\sin x}{\cos x} \, dx &= \int \frac{\sin x}{u} \frac{du}{-\sin x} = - \int \frac{1}{u} \, du \\ &= -\ln|u| + C = -\ln|\cos x| + C \end{aligned}$$

Note that

$$-\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$$

Thus we can also write:

$$\int \tan x \, dx = \ln|\sec x| + C$$

The Substitution Rule for Definite Integrals

Methods for evaluating a **definite integral using substitution**:

$$\int_a^b f(x) dx$$

Method 1

- ▶ evaluate the indefinite integral $\int f(x) dx$ using substitution
- ▶ then use the Fundamental Theorem $\int_a^b f(x) dx = \int f(x) dx \Big|_a^b$

$$\int_0^4 \sqrt{2x+1} dx = \int \sqrt{2x+1} dx \Big|_0^4 = \frac{1}{3} (2x+1)^{\frac{3}{2}} \Big|_0^4 = 9 - \frac{1}{3} = \frac{26}{3}$$

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$$\int_0^4 \sqrt{2x+1} dx$$

We choose $u = 2x + 1$. Then $u' = 2$, and hence

$$\begin{aligned}\int_0^4 \sqrt{2x+1} dx &= \int_{u(0)}^{u(4)} \sqrt{u} \frac{du}{2} = \frac{1}{2} \int_1^9 \sqrt{u} du \\ &= \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^9 = \frac{1}{2} \left(\frac{2}{3} 9^{3/2} - \frac{2}{3} 1^{3/2} \right) \\ &= \frac{1}{2} \left(\frac{2}{3} \sqrt{9^3} - \frac{2}{3} \right) = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}\end{aligned}$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$$\int_1^2 \frac{1}{(3-5x)^2} dx$$

We choose $u = 3 - 5x$. Then $u' = -5$, and hence

$$\begin{aligned} \int_1^2 \frac{1}{(3-5x)^2} dx &= \int_{u(1)}^{u(2)} \frac{1}{u^2} \frac{du}{-5} = -\frac{1}{5} \int_{-2}^{-7} \frac{1}{u^2} du \\ &= -\frac{1}{5} \left(-\frac{1}{u} \right) \Big|_{-2}^{-7} = -\frac{1}{5} \left(-\frac{1}{-7} - \left(-\frac{1}{-2} \right) \right) \\ &= -\frac{1}{5} \left(\frac{1}{7} - \frac{1}{2} \right) = -\frac{1}{5} \left(\frac{2}{14} - \frac{7}{14} \right) = \frac{1}{14} \end{aligned}$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$$\int_1^e \frac{\ln x}{x} dx$$

We choose $u = \ln x$. Then $u' = \frac{1}{x}$, and hence

$$\begin{aligned} \int_1^e \frac{\ln x}{x} dx &= \int_{u(1)}^{u(e)} \frac{u}{x} \frac{du}{1/x} = \int_0^1 u du \\ &= \left(\frac{1}{2} u^2 \right) \Big|_0^1 = \frac{1}{2} 1^2 - \frac{1}{2} 0^2 = \frac{1}{2} \end{aligned}$$

The Substitution Rule: Exercises

$$\int e^{-x} dx$$

take $u = -x$

$$\int x^3(2 + x^4)^5 dx$$

take $u = 2 + x^4$

$$\int x^2 \sqrt{x^3 + 1} dx$$

take $u = x^3 + 1$

$$\int \frac{1}{(1 - 6t)^4} dt$$

take $u = 1 - 6t$

$$\int \cos^3 \phi \sin \phi d\phi$$

take $u = \cos \phi$

$$\int \frac{\sec^2(\frac{1}{x})}{x^2} dt$$

take $u = \frac{1}{x}$

The Substitution Rule: Exercises

$$\int (x+1)\sqrt{2x+x^2} dx \quad \text{take } u = 2x+x^2, \text{ then } u' = 2(1+x)$$

$$\int (3t+2)^{2.4} dt \quad \text{take } u = 3t+2$$

$$\int e^x \cos e^x dx \quad \text{take } u = e^x$$

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx \quad \text{take } u = \sqrt{x}, \text{ then } u' = \frac{1}{2\sqrt{x}}$$

$$\int \frac{(\ln x)^2}{x} dx \quad \text{take } u = \ln x$$

$$\int (x^3+3x)(x^2+1) dx \quad \text{take } u = x^3+3x, \text{ then } u' = 3(x^2+1)$$

The Substitution Rule: Exercises

Evaluate

$$\int x \sin(x^2) dx$$

We take $u = x^2$, then $u' = 2x$ and

$$\begin{aligned}\int x \sin(x^2) dx &= \int x \sin u \frac{du}{2x} \\ &= \frac{1}{2} \int \sin u du \\ &= -\frac{1}{2} \cos u + C \\ &= -\frac{1}{2} \cos x^2 + C\end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int x^2 e^{x^3} dx$$

We take $u = x^3$, then $u' = 3x^2$ and

$$\begin{aligned}\int x^2 e^{x^3} dx &= \int x^2 e^u \frac{du}{3x^2} \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{x^3} + C\end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int \frac{\sin 2x}{1 + \cos^2 x} dx$$

We recall that $\sin 2x = 2 \sin x \cos x$

We take $u = 1 + (\cos x)^2$.

Then $u' = 2 \cos x(-\sin x) = -\sin 2x$ and

$$\begin{aligned} \int \frac{\sin 2x}{1 + \cos^2 x} dx &= \int \frac{\sin 2x}{u} \frac{du}{-\sin 2x} \\ &= - \int \frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln |1 + (\cos x)^2| + C \end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int_0^1 \cos(\pi x/2) dx$$

We take $u = \pi x/2$, then $u' = \pi/2$ and

$$\begin{aligned} \int_0^1 \cos(\pi x/2) dx &= \int_{u(0)}^{u(1)} \cos(u) \frac{du}{\pi/2} \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos(u) du \\ &= \frac{2}{\pi} \left(\sin u \Big|_0^{\pi/2} \right) \\ &= \frac{2}{\pi} (\sin(\pi/2) - \sin(0)) \\ &= \frac{2}{\pi} \end{aligned}$$

Symmetry

Symmetry can sometimes help to simplify integrals!

Suppose f is continuous on $[-a, a]$:

- ▶ If f is even [$f(-x) = f(x)$], then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

- ▶ If f is odd [$f(-x) = -f(x)$], then

$$\int_{-a}^a f(x) dx = 0$$

$$\int_{-1}^1 f(x) dx = 0 \quad \text{where} \quad f(x) = \frac{\tan x}{1 + x^2 + x^4}$$

The function f is **odd** since

$$f(-x) = \frac{\sin(-x)/\cos(-x)}{1 + (-x)^2 + (-x)^4} = \frac{-\sin x/\cos x}{1 + x^2 + x^4} = -f(x)$$