Calculus M211

Jörg Endrullis

Indiana University Bloomington

2013

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus establishes a connection between:

- differentiation calculus, and
- integration calculus

Differentiation and integration are inverse processes!

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

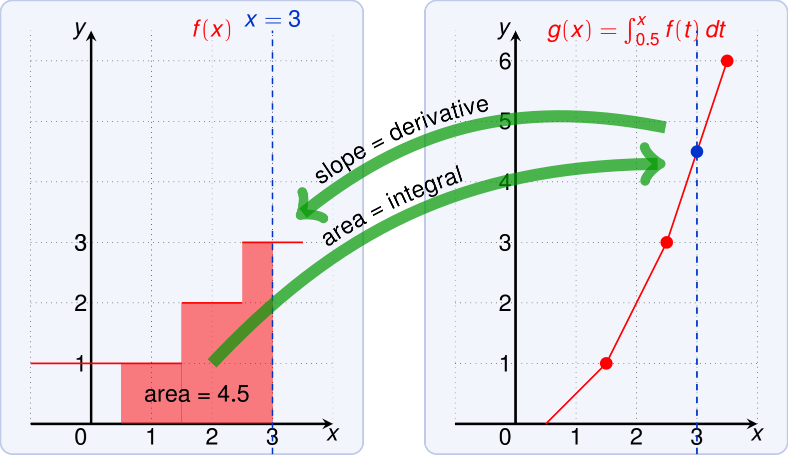
Suppose f is a continuous function on $[a, b]$. Then

1. If
$$g(x) = \int_a^x f(t)dt$$
then $g'(x) = f(x)$.
2. Let F be any antiderivative of f, that is, $F' = f$. Then
$$\int_a^b f(x)dx = F(b) - F(a)$$

The first part of the theorem can be written as:
$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
The second part can be written as:
$$\int_a^b F'(x)dx = F(b) - F(a)$$

Fundamental Theorem of Calculus

$$g(x) = \int_a^x f(t)dt \qquad \Longrightarrow \qquad g'(x) = f(x)$$



Observe: $g'(x) = f(x)$ except where f is not continuous.

The slope (derivative) is the inverse of taking the area (integral).

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

$\int_0^x \sin x \, dx = F(x) - F(0) = -\cos x - (-\cos 0) = 1 - \cos x$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Find the derivative of

$$g(x) = \int_0^x \sqrt{1+t^2}dt$$

By the Fundamental Theorem of Calculus, part 1:

$$g'(x) = \sqrt{1+x^2}$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Find

$$g(x) = \frac{d}{dx} \int_1^{x^4} \sec t \, dt$$

Lets introduce a name for the integral without x^4:

$$f(x) = \int_1^x \sec t \, dt \qquad\qquad f'(x) = \sec x$$

Then

$$g(x) = \frac{d}{dx} f(x^4) = f'(x^4) \cdot 4x^3 = \sec(x^4) \cdot 4x^3$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

The second part yields an easy method for evaluating integrals!

Evaluate the integral

$$\int_1^3 e^x\, dx$$

Note that e^x is continuous, and an antiderivative is $F(x) = e^x$.

$$\int_1^3 e^x\, dx = e^3 - e$$

We could have used any antiderivative $F(x) = e^x + C$!

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

We often use the notation:

$$F(x)\big]_a^b = F(b) - F(a)$$

Then

$$\int_a^b f(x)dx = F(x)\big]_a^b$$

Alternative notation

$$F(x)\Big|_a^b \quad = \quad \Big[F(x)\Big]_a^b \quad = \quad F(x)\Big]_a^b$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Find the area under the parabola

$$f(x) = x^2$$

from 0 to 1.

From 0 to 1 the curve is above the x-axis. Thus area = integral.

An antiderivative of f is $F(x) = \frac{1}{3}x^3$.

By the Fundamental Theorem, the area is:

$$A = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Evaluate

$$\int_3^6 \frac{1}{x} dx$$

An antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$. Then

$$\int_3^6 \frac{1}{x} dx = \ln|x|\Big]_3^6 = \ln 6 - \ln 3 = \ln \frac{6}{3} = \ln 2$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Find the area under $\cos x$ from 0 to b where $0 \leq b \leq \pi/2$.

For $0 \leq x \leq \pi/2$, we have $\cos x \geq 0$. Thus area = integral.

An antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$. Then

$$\int_0^b \cos x \, dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Suppose f is a continuous function on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Evaluate

$$\int_{-1}^{3} \frac{1}{x^2} dx \quad \text{does not exist}$$

An antiderivative of $f(x) = \frac{1}{x^2}$ is $F(x) = -\frac{1}{x}$. Then

$$\int_{-1}^{3} \frac{1}{x^2} dx = -\frac{1}{x}\Big]_{-1}^{3} = -\frac{1}{3} - (-\frac{1}{-1}) = -\frac{4}{3}$$

Does this make sense? Note that $\frac{1}{x^2}$ is above the x-axis!

The calculation is wrong since $\frac{1}{x^2}$ is not continuous on $[-1, 3]$!