

# Calculus M211

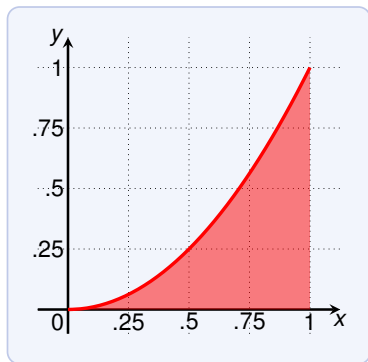
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# The Area below a Curve

How to compute the area below a curve?

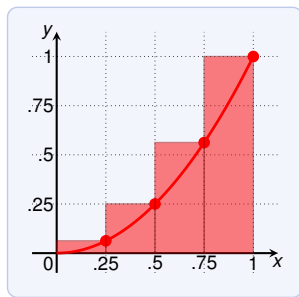


Idea:

- ▶ divide the area in vertical strips of equal width
- ▶ approximate the area using rectangles

# The Area below a Curve

Estimate the area below the curve  $f(x) = x^2$  from 0 to 1.



Let's split the area in 4 vertical strips:

- ▶ height of each rectangle = value at right endpoint

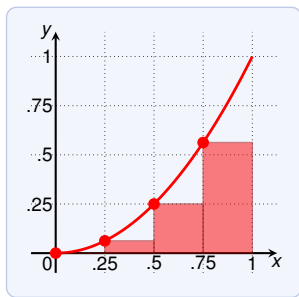
The sum of the area of these rectangles is:

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = 0.46875$$

The area  $A$  below the curve is less than  $R_4$ , that is,  $A < R_4$ .

# The Area below a Curve

Estimate the area below the curve  $f(x) = x^2$  from 0 to 1.



Let's split the area in 4 vertical strips:

- ▶ height of each rectangle = value at **left** endpoint

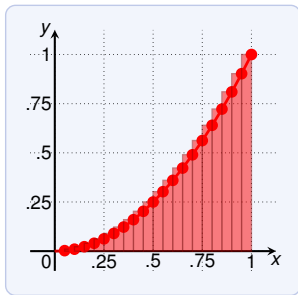
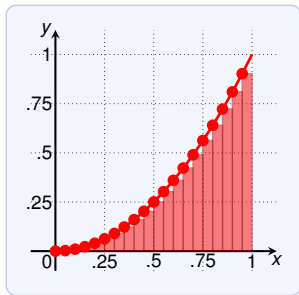
The sum of the area of these rectangles is:

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = 0.21875$$

The area  $A$  below the curve is larger than  $L_4$ , that is,  $L_4 < A$ .

# The Area below a Curve

Estimate the area below the curve  $f(x) = x^2$  from 0 to 1.



$$0.21875 = L_4 < A < R_4 = 0.46875$$

$$0.2734375 = L_8 < A < R_8 = 0.3984375$$

$$0.3087500 = L_{20} < A < R_{20} = 0.3587500$$

We have obtained an estimation of  $A$ :

- we can improve the estimation by taking more strips

# The Area below a Curve

Estimate the area below the curve  $f(x) = x^2$  from 0 to 1.

We now let the number of strips go to infinity:  $\lim_{n \rightarrow \infty} R_n$

The formula for the area  $R_n$  with  $n$  strips is:

$$\begin{aligned} R_n &= \frac{1}{n} \cdot \left(\frac{1}{n}\right)^2 + \frac{1}{n} \cdot \left(\frac{2}{n}\right)^2 + \frac{1}{n} \cdot \left(\frac{3}{n}\right)^2 + \dots + \frac{1}{n} \cdot \left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n} \cdot \frac{1}{n^2} (1^2 + 2^2 + 3^2 + \dots + n^2) \end{aligned}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Hence, we have

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{1}{6} \left(\frac{n+1}{n}\right) \left(\frac{2n+1}{n}\right) = \frac{2}{6} = \frac{1}{3}$$

# The Area below a Curve

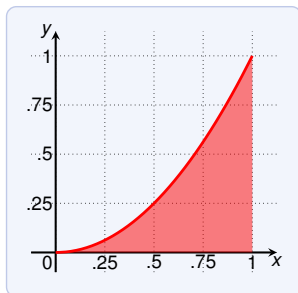
Estimate the area below the curve  $f(x) = x^2$  from 0 to 1.

$$\lim_{n \rightarrow \infty} R_n = \frac{1}{3}$$

and similar

$$\lim_{n \rightarrow \infty} L_n = \frac{1}{3}$$

The right- and left-approximations converge to the same value.



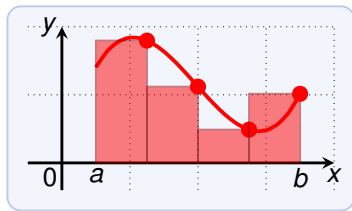
We define the area  $A$  to be the limit of these approximations

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n = \frac{1}{3}$$

# The Area below a Curve

Now let's look at a general curve above the  $x$ -axis:

The area below the curve of a function  $f$  on an interval  $[a, b]$ .



We use  $n$  rectangles:

- ▶ the width of the interval is  $b - a$
- ▶ the width of each strip is  $\Delta x = (b - a)/n$
- ▶ the interval for the  $i$ -th strip is  $I_i = [a + (i - 1)\Delta x, a + i\Delta x]$



# The Area below a Curve

Now let's look at a general curve above the  $x$ -axis:

The area below the curve of a function  $f$  on an interval  $[a, b]$ .



We use  $n$  rectangles:  $\Delta x = (b - a)/n$

The area of the rectangles oriented at right-endpoints is:

$$\begin{aligned} R_n &= \Delta x \cdot f(a + 1\Delta x) + \Delta x \cdot f(a + 2\Delta x) + \dots + \Delta x \cdot f(a + n\Delta x) \\ &= \Delta x (f(a + 1\Delta x) + f(a + 2\Delta x) + \dots + f(a + n\Delta x)) \end{aligned}$$

The area of the rectangles oriented at left-endpoints is:

$$\begin{aligned} L_n &= \Delta x \cdot f(a + 0\Delta x) + \Delta x \cdot f(a + 1\Delta x) + \dots + \Delta x \cdot f(a + (n-1)\Delta x) \\ &= \Delta x (f(a + 0\Delta x) + f(a + 1\Delta x) + \dots + f(a + (n-1)\Delta x)) \end{aligned}$$

# The Area below a Curve

The **area**  $A$  under the graph of a continuous function  $f$  whose graph lies **above the  $x$ -axis** is the limit:

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} R_n \\ &= \lim_{n \rightarrow \infty} [\Delta x (f(a + 1\Delta x) + f(a + 2\Delta x) + \dots + f(a + n\Delta x))] \end{aligned}$$

where  $\Delta x = (b - a)/n$ .



For continuous  $f$  this limit always exists, and is the same as

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} [\Delta x (f(a + 0\Delta x) + \dots + f(a + (n - 1)\Delta x))] ]$$

# The Area below a Curve

Recall that the interval of the  $i$ -th strip is:

$$I_i = [a + (i - 1)\Delta x, a + i\Delta x]$$

$$R_n = \Delta x (f(x_1) + f(x_2) + \dots + f(x_n))$$

where  $x_i$  is the right endpoint of the interval  $I_i$

$$L_n = \Delta x (f(x_1) + f(x_2) + \dots + f(x_n))$$

where  $x_i$  is the left endpoint of the interval  $I_i$

For continuous curve  $f$  above the  $x$ -axis we have:

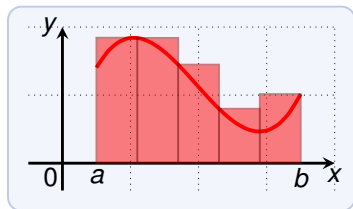
$$A = \lim_{n \rightarrow \infty} [\Delta x (f(x_1) + f(x_2) + \dots + f(x_n))]$$

independent of what **sample points**  $x_i$  we take from  $I_i$ .

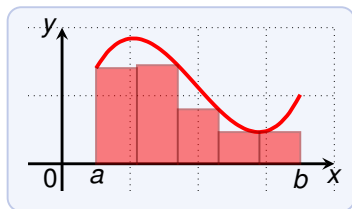
The limit is the same no matter what  $x_i$  we choose from  $I_i$ !

A famous choice of sample points are upper and lower sums. . .

# The Area below a Curve



upper sum  $U_5$



lower sum  $D_5$

The **upper sum** is

$$U_n = \Delta x (f(x_1) + f(x_2) + \dots + f(x_n))$$

where  $x_i$  is chosen from  $I_i$  such that  $f(x_i)$  is the maximum on  $I_i$

The **lower sum** is

$$D_n = \Delta x (f(x_1) + f(x_2) + \dots + f(x_n))$$

where  $x_i$  is chosen from  $I_i$  such that  $f(x_i)$  is the minimum on  $I_i$

# Sigma Notation

We can use the **sigma notation** to write sums more compactly:

$$\sum_{i=1}^n h(i) = h(1) + h(2) + \dots + h(n)$$

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

# The Area below a Curve

The area under a curve  $f$  above the  $x$ -axis from  $a$  to  $b$  is:

$$A = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \Delta x \cdot f(x_i) \right)$$

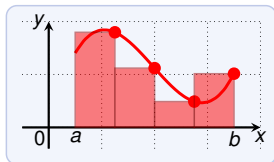
where:

- ▶  $\Delta x = (b - a)/n$  is the width of the strips,
- ▶  $I_i = [a + (i - 1)\Delta x, a + i\Delta x]$  is the interval of the  $i$ -th strip,
- ▶  $x_i$  is the sample point from the  $i$ -th interval  $I_i$ .

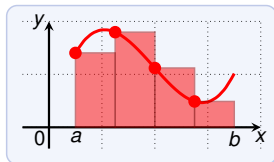
Usual choices for  $x_i$  are

- ▶ left endpoint  $x_i = a + (i - 1)\Delta x$  of the interval
- ▶ right endpoint  $x_i = a + i\Delta x$  of the interval
- ▶ middle  $x_i = a + (i - \frac{1}{2})\Delta x$  of the interval
- ▶ upper sum:  $f(x_i)$  is the maximum on the interval  $I_i$
- ▶ lower sum:  $f(x_i)$  is the minimum on the interval  $I_i$

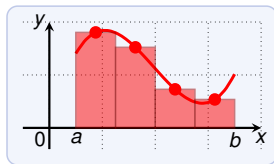
# The Area below a Curve



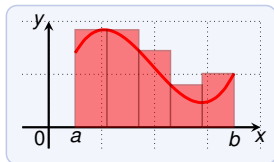
right-endpoints with 5 strips



left-endpoints with 5 strips



midpoints with 5 strips



upper sum with 5 strips



lower sum with 5 strips