Calculus M211

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Antiderivatives / Integrals

A function F is called **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Let $f(x) = x^2$ then an antiderivative of f is

$$F(x) = \frac{1}{3}x^3$$

However f has more antiderivatives; every function of the form

$$G(x) = \frac{1}{3}x^3 + C \qquad \text{where } C \text{ is a constant}$$

Can there be other antiderivatives? No! by next theorem. . .

Antiderivatives / Integrals

If F is an antiderivative of f on an interval I,
then the **most general antiderivative** of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Find the general antiderivatives of the following functions:

- $f(x) = \sin x$

  $F(x) = -\cos x + C$

- $g(x) = x^n \quad \text{for } n \neq -1$

  $G(x) = \frac{1}{n+1} x^{n+1} + C$

  If $n \geq 0$, then this is valid for any interval.
  If $n < 0$ & $n \neq -1$, then valid for intervals not containing 0.

Antiderivatives / Integrals

Find the general antiderivatives of

$$f(x) = \frac{1}{x}$$

We have

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Thus $\ln x + C$ is the general antiderivative on $(0, \infty)$.

We moreover know that:

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

Thus $\ln |x| + C$ is the general antiderivative on intervals not containing 0. In particular on intervals $(-\infty, 0)$ and $(0, \infty)$.

So the general antiderivative of f is:

$$F(x) = \begin{cases} \ln x + C_1 & \text{for } x > 0 \\ \ln(-x) + C_2 & \text{for } x < 0 \end{cases}$$

Antiderivatives / Integrals

Let $F' = f$ and $G' = g$.

The following table gives examples of particular antiderivatives:

Function	Antiderivative
$c\,f(x)$	$c\,F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
x^n, $n \neq -1$	$(x^{n+1})/(n+1)$
$1/x$	$\ln\|x\|$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Antiderivatives / Integrals

Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

We first simplify

$$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$$

Then the general antiderivative of g' is:

$$g(x) = 4(-\cos x) + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

In applications of calculus, finding antiderivatives is common:

- we measure the speed, and want the distance traveled
- we measure the acceleration, and wand to know the speed
- ...

Antiderivatives / Integrals

Find f if

$$f''(x) = 12x^2 + 6x - 4$$

and $f(0) = 4$ and $f(1) = 1$.

The general antiderivative of f'' is:

$$f'(x) = 4x^3 + 3x^2 - 4x + C$$

The general antiderivative of f' is:

$$f(x) = x^4 + x^3 - 2x^2 + Cx + D$$

To ensure $f(0) = 4$ and $f(1) = 1$, we need to find C and D:

$$f(0) = D = 4$$
$$f(1) = 1 + 1 - 2 + C + 4 = C + 4 = 1 \quad \Longrightarrow \quad C = -3$$

Therefore the function f we are looking for is:

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

Antiderivatives / Integrals

A particle moves in a straight line and has

- acceleration $a(t) = 6t + 4$
- initial velocity is $v(0) = -6$cm/s
- initial displacement is $s(0) = 9$cm

Find the position function $s(t)$.

The velocity is an antiderivative of the acceleration:

$$v(t) = 3t^2 + 4t + C$$

As $v(0) = -6$cm/s, it follows that $C = -6$.

The position function is an antiderivative of the velocity:

$$s(t) = t^3 + 2t^2 - 6t + D$$

As $s(0) = 9$cm/s, it follows that $D = 9$.

Thus the position function is:

$$s(t) = t^3 + 2t^2 - 6t + 9 \quad \text{in cm}$$

Antiderivatives / Integrals

Near the surface of the earth, the gravitational force produces a downward acceleration of approximately 9.8m/s^2 (or 32ft/s^2).

A ball is thrown upward with a speed of 48ft/s from the edge of cliff 432ft above ground. When does the ball reach its maximum height? When does it hit the ground?

Let $s(t)$ be the distance above the ground, and $v(t)$ the velocity:

$$\begin{aligned} a(t) &= -32 \\ v(t) &= -32t + C \qquad & v(0) = C = 48 \\ s(t) &= -16t^2 + 48t + D \qquad & s(0) = D = 432 \end{aligned}$$

The ball reaches the maximal height when

$$v(t) = 0 = -32t + 48, \text{ that is, after } t = 1.5 \text{ seconds}$$

The ball hits the ground when

$$s(t) = 0 = -16t^2 + 48t + 432 \iff t^2 - 3t - 27 = 0$$

We reject the negative solution, and find $t = 3/2 + 3/2 \cdot \sqrt{13}$.