Calculus M211

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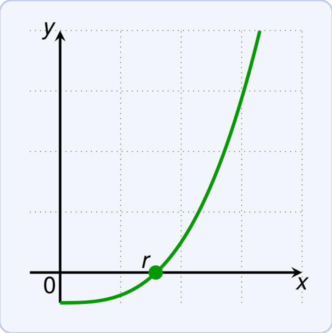
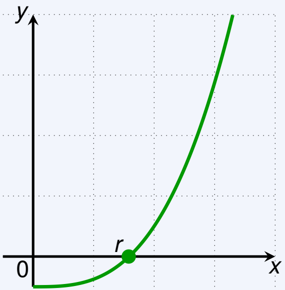
Indiana University Bloomington

2013

Newton's Method



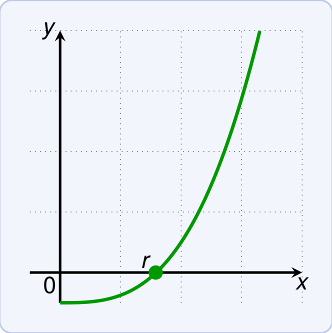
Newton's Method



Assume we want to find a root of a complicated function like:

$$f(x) = x^7 - x + \cos x$$

Newton's Method



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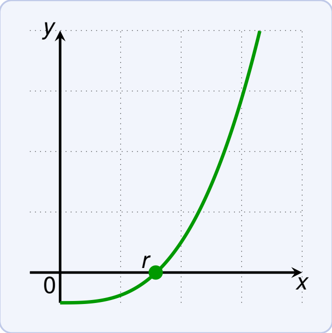

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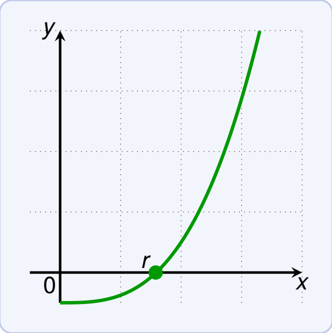

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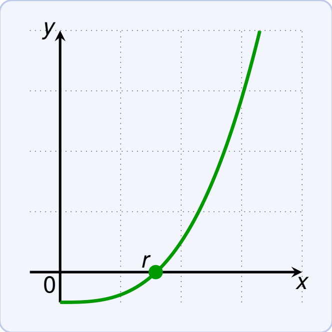
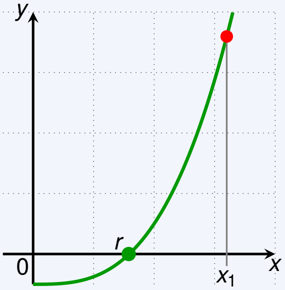
Can we at least find the root approximately?

Newton's Method



Idea of Newton's Method

Newton's Method



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- Take an approximation x_1 of the root (a rough guess).

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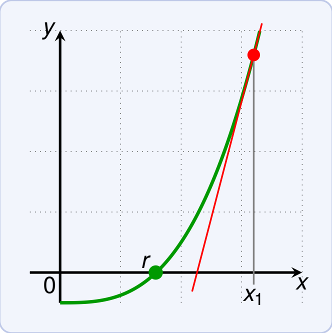

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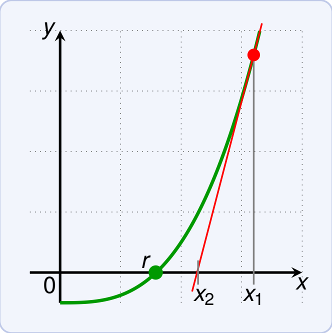

Idea of Newton's Method

- Take an approximation x_1 of the root (a rough guess).
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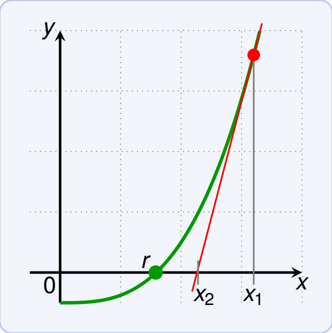

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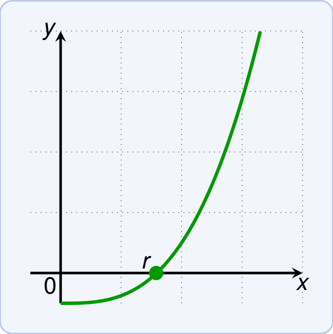
We can repeat this procedure to get improve the approximation.

Newton's Method



We want to find an approximation of the root r of $f(x)$.

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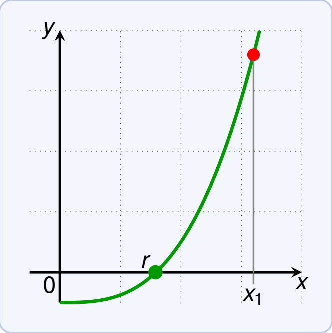

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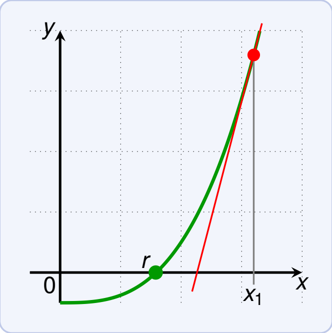

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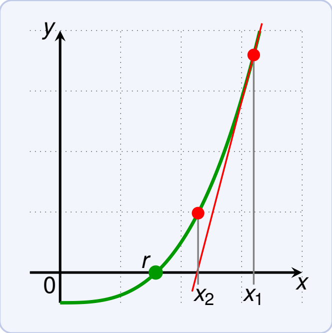

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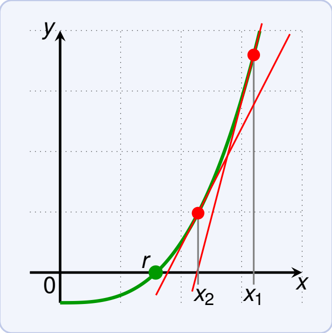

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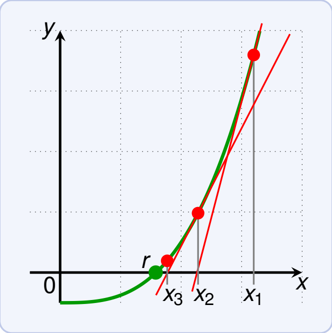

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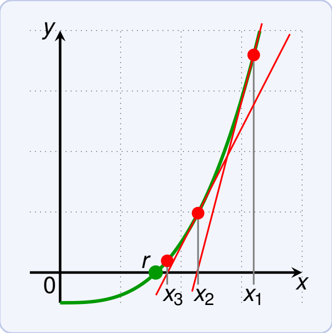

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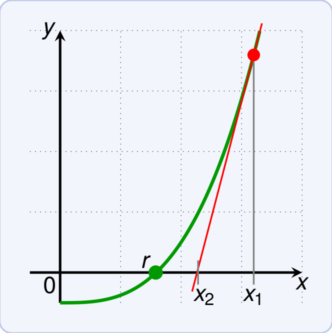
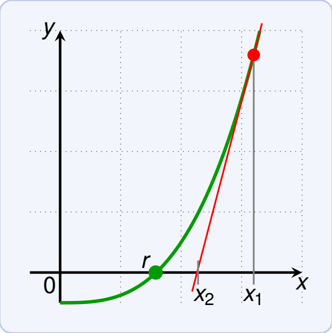
- ...continue until approximation is good enough

Newton's Method



How can we compute x_2?

Newton's Method



How can we compute x_2? The tangent at $(x_1, f(x_1))$ is

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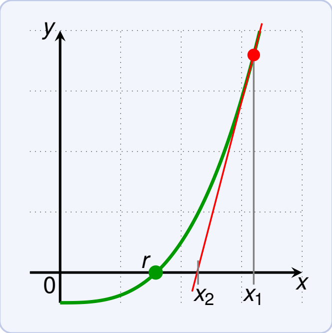
Newton's Method



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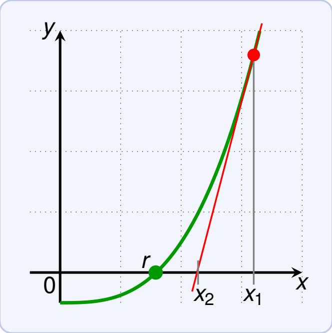

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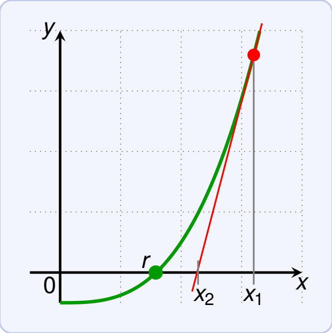

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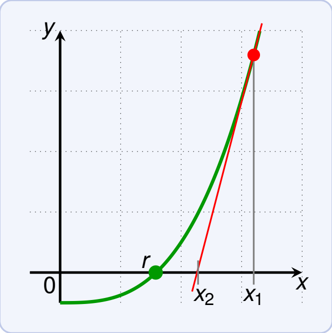

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We can repeat this process to get x_3, x_4, x_5...

Newton's Method

Let $f(x)$ be a function, and x_1 and approximation of a root r.

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Let $f(x)$ be a function, and x_1 and approximation of a root r.

We compute a sequence $x_2, x_3, x_4, \ldots$ of approximations by

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The sequence $x_1, x_2, x_3, \ldots$ gets closer and closer to the root 1.

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The hope is that $x_2, x_3, \ldots$ get closer and closer to the root r.
However, this does not always work.

Let $x_1 = 1$. Find the 2nd approximation to the root of $\sqrt[3]{x}$.

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The hope is that $x_2, x_3, \ldots$ get closer and closer to the root r.
However, this does not always work.

Let $x_1 = 1$. Find the 2nd approximation to the root of $\sqrt[3]{x}$.

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{\left(\frac{1}{3}\right)} = -2$$

Newton's Method

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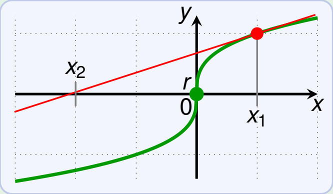

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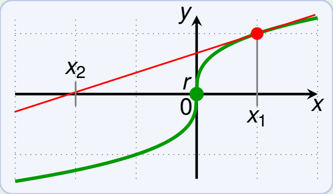

Note that $x_2 = -2$ is further away from the root 0 than $x_1 = 1$.

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For more complicated examples see

- Chapter 4.8, Examples 1,2 and 3