Calculus M211

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Optimization

We now use calculus to solve practical problems.

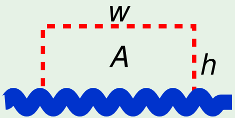
Challenge: convert word problems into mathematical problems

- understand the problem
- draw a diagram
- introduce notation
- translate the problem to the notation
- use calculus to solve it

Optimization

A farmer has 2400ft of fencing and wants to fence a rectangular field that borders a straight river. No fence needed along river.

What are the dimensions of the field with the largest area?



Introducing notation:

- let h be the height of the field
- let w be the width (parallel to river)
- let A be the area

What do we know?

$$2400 = 2h + w \implies w = 2400 - 2h \quad \text{for } h \text{ in } [0, 1200]$$

$$A = hw = h(2400 - 2h) = 2400h - 2h^2 \quad \text{for } h \text{ in } [0, 1200]$$

A is continuous on $[0, 1200]$, we use the Closed Interval Method:

$$A'(h) = 2400 - 4h \qquad A'(h) = 0 \iff h = 2400/4 = 600$$

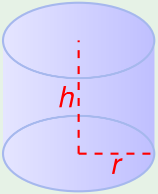
The value of A at critical number 600 and the interval ends are:

$$A(0) = 0 \qquad A(600) = 600 \cdot 1200 \qquad A(1200) = 0$$

The dimensions of the field are: 600ft height, 1200ft width.

Optimization

A cylindrical can is made to hold 1L of oil. Find the dimensions that minimize the cost of the metal to manufacture the can.



Introducing notation:

- let h be the height
- let r be the radius
- let V be the volume
- let A be the surface area

$$V = \pi r^2 h = 1 \implies h = 1/(\pi r^2)$$
$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2/r \qquad \text{for } r \text{ in } (0, \infty)$$
$$A'(r) = 4\pi r - 2/r^2 = (4\pi r^3 - 2)/r^2$$
$$A'(r) = 0 \iff r = 1/\sqrt[3]{2\pi} \text{ is the only critical number}$$

Cannot use Closed Interval Method since $(0, \infty)$ is not closed.

However, $A(1/\sqrt[3]{2\pi})$ must be the **absolute minimum** since:

- A is decreasing, $A'(r) < 0$, for all $r < 1/\sqrt[3]{2\pi}$,
- A is increasing, $A'(r) > 0$, for all $r > 1/\sqrt[3]{2\pi}$.

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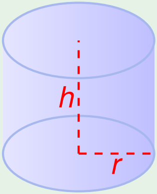

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However, $A(1/\sqrt[3]{2\pi})$ must be the **absolute minimum**

Then $h = 1/(\pi r^2) = \sqrt[3]{2\pi}^2/\pi = \sqrt[3]{4\pi^2/\pi^3} = 2/\sqrt[3]{2\pi} = 2r$

Hence radius $r = 1/\sqrt[3]{2\pi}$ and height $h = 2r$ minimizes the cost.

The argument we have used on the last slide is the following:

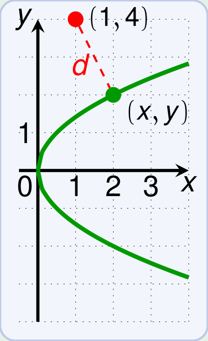
First Derivative Test for Absolute Extreme Values

Let f be continuous, defined on an open or closed interval.
Let c be a critical number of f.

- If $f'(x) > 0$ for all $x < c$, and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum of f.
- If $f'(x) < 0$ for all $x < c$, and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum of f.

Optimization

Find the point on the parabola $y^2 = 2x$ that is closest to $(1,4)$.



Introducing notation:

- let d be the distance of (x,y) to $(1,4)$

Then

$$d = \sqrt{(x-1)^2 + (y-4)^2} \qquad x = y^2/2$$

Square root makes derivative complicated.
Note that d minimal $\iff$ d^2 minimal.
Thus, instead of d we minimize d^2!

$$f(y) = d^2 = (y^2/2 - 1)^2 + (y-4)^2$$

$$f'(y) = 2(y^2/2 - 1)y + 2(y-4) = y^3 - 8$$

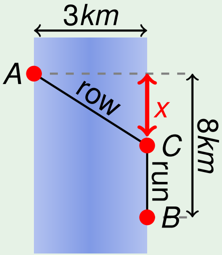
$$f'(y) = 0 \iff y = 2$$

Moreover $f'(y) < 0$ for all $y < 2$ and $f'(y) > 0$ for all $y > 2$.

Thus by the First Derivative Test for Absolute Extrema, $f(2)$ is the absolute minimum. Thus the point $(2,2)$ is closest to $(1,4)$.

Optimization

A man wants wants to get from point A on one side of a 3km wide river to point B, 8km downstream on the opposite side. He can row 6km/h and run 8km/h. Where to land to be fastest?



Introducing notation:

- let C be the landing point
- let x = downstream distance of A to C

The time for rowing is and running:

$$t_{\text{row}}(x) = (\sqrt{3^2 + x^2})/6$$

$$t_{\text{run}}(x) = (8 - x)/8$$

The total time is $\quad t(x) = t_{\text{row}}(x) + t_{\text{run}}(x) \quad$ for x in $[0, 8]$

$$t'(x) = \frac{x}{6\sqrt{3^2 + x^2}} - \frac{1}{8} \qquad t'(x) = 0 \iff x = 9/\sqrt{7}$$

$$t'(x) = 0 \iff 3\sqrt{3^2 + x^2} = 4x \overset{x \geq 0}{\iff} 9(3^2 + x^2) = 16x^2$$

$$\iff 7x^2 = 81 \iff x^2 = 81/7 \overset{x \geq 0}{\iff} x = 9/\sqrt{7}$$

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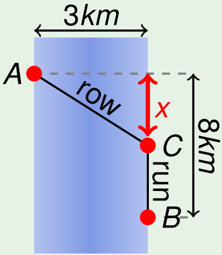

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The total time is $t(x) = t_{\text{row}}(x) + t_{\text{run}}(x)$ for x in $[0, 8]$

$$t'(x) = \frac{x}{6\sqrt{3^2 + x^2}} - \frac{1}{8} \qquad t'(x) = 0 \iff x = 9/\sqrt{7}$$

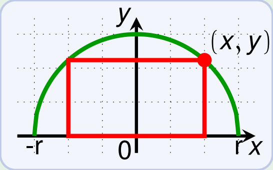
Now we apply the Closed Interval Method:

$$t(0) = 1.5 \quad t(9/\sqrt{7}) = 1 + \sqrt{7}/8 \approx 1.33 \quad t(8) = \sqrt{73}/6 \approx 1.42$$

Thus landing $9/\sqrt{7}$km downstream is the fastest.

Optimization

Find the area of the largest rectangle that can be inscribed in a semi-circle circle of radius r.



Introducing notation:

- let (x, y) be the upper right corner of the rectangle
- let A be the area

The area is $\quad A(x) = 2xy = 2x\sqrt{r^2 - x^2} \quad$ for x in $[0, r]$

A is continuous on $[0, r]$, we use the Closed Interval Method:

$$A'(x) = 2\sqrt{r^2 - x^2} + \frac{2x}{2\sqrt{r^2 - x^2}}(-2x) = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

$$A'(x) = 0 \iff x^2 = r^2/2 \overset{x \geq 0}{\iff} x = r/\sqrt{2}$$

Note that $A(0) = 0$ and $A(r) = 0$. Thus the maximum area is:

$$A(r/\sqrt{2}) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{\sqrt{2}^2}} = \sqrt{2}r\sqrt{\frac{r^2}{2}} = r^2$$

Optimization

A store sells 100 blu-ray players per week for 200$ each. A market survey shows that for each 10$ discount, the store would sell 40 more players per week. The store buys the players at a price of 150$ per piece.

What selling price would maximize the profit of the store?

Introducing notation:

- let x be the discount
- let s be the number of players sold, and p the profit

$$s(x) = 100 + 40 \cdot \frac{x}{10} = 100 + 4x$$
$$p(x) = s(x) \cdot (200 - x - 150) = (100 + 4x) \cdot (50 - x)$$
$$= -4x^2 + 100x + 5000 \qquad \text{for } x \text{ in } [0, 50]$$
$$p'(x) = -8x + 100 \qquad p'(x) = 0 \iff x = 12.5$$

Note that $p(x)$ is continuous, and

$$p(0) = 5000 \qquad p(12.5) = 5625 \qquad p(50) = 0$$

By the Closed Interval Method, 12.5$ discount for maximal profit.

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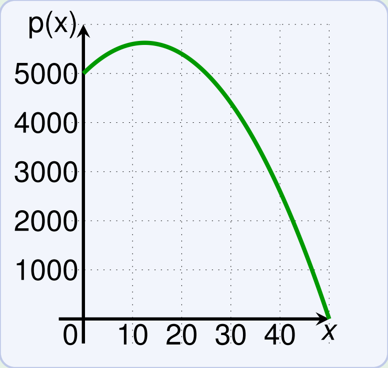

By the Closed Interval Method, 12.5$ discount for maximal profit.