

# Calculus M211

Jörg Endrullis

Indiana University Bloomington

2013

# Curve Sketching

For sketching a curve of  $f(x)$ :

- ▶ determine the **domain**
- ▶ find the **y-intercept**  $f(0)$  and the **x-intercepts**  $f(x) = 0$
- ▶ find **vertical asymptotes**  $x = a$ , that is:

$$\lim_{x \rightarrow a^-} = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} = \pm\infty$$

- ▶ find **horizontal asymptotes**  $y = L$ , that is:

$$\lim_{x \rightarrow \infty} = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} = L$$

- ▶ find intervals of **increase**  $f'(x) > 0$  and **decrease**  $f'(x) < 0$
- ▶ find **local maxima and minima**
- ▶ determine **concavity** on intervals and **points of inflection**
  - ▶  $f''(x) > 0$  concave upward
  - ▶  $f''(x) < 0$  concave downward
  - ▶ inflections points where  $f''(x)$  changes the sign

# Curve Sketching

For local minima and maxima:

- ▶ find critical numbers  $c$
- ▶ then the first First Derivative Test:
  - ▶  $f'$  changes from  $+$  to  $-$  at  $c \implies$  maximum
  - ▶  $f'$  changes from  $-$  to  $+$  at  $c \implies$  minimum
- ▶ Second Derivative Test:
  - ▶  $f''(c) < 0 \implies$  maximum
  - ▶  $f''(c) > 0 \implies$  minimum
  - ▶  $f''(c) = 0 \implies$  use First Derivative Test

Then sketch the curve:

- ▶ draw asymptotes as thin dashed lines
- ▶ mark intercepts, local extrema and inflection points
- ▶ draw the curve taking into account:
  - ▶ increase / decrease, concavity and asymptotes

# Curve Sketching

Sketch the curve of  $f(x) = \frac{2x^2}{x^2-1}$ .

The domain is  $\{x \mid x \neq \pm 1\}$ , that is,  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

We have  $f(0) = 0$  and  $f(x) = 0 \iff x = 0$

The vertical asymptotes are  $x = -1$  and  $x = 1$

$$\lim_{x \rightarrow -1^-} f(x) = \infty \quad \lim_{x \rightarrow -1^+} f(x) = -\infty \quad \lim_{x \rightarrow 1^-} f(x) = -\infty \quad \lim_{x \rightarrow 1^+} f(x) = \infty$$

The horizontal asymptotes are  $y = 2$

$$\lim_{x \rightarrow \infty} f(x) = 2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

# Curve Sketching

Sketch the curve of  $f(x) = \frac{2x^2}{x^2-1}$ .

The derivative is:

$$f'(x) = \frac{4x(x^2 - 1) - 2x^2(2x)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Thus

- ▶ increasing ( $f'(x) > 0$ ) on  $(-\infty, -1) \cup (-1, 0)$
- ▶ decreasing on ( $f'(x) < 0$ ) on  $(0, 1) \cup (1, \infty)$

The critical numbers are  $x = 0$  (since  $f'(0) = 0$ )

- ▶  $f'(x)$  changes from  $+$  to  $-$  at  $0 \implies$  local maximum  $(0, 0)$

# Curve Sketching

Sketch the curve of  $f(x) = \frac{2x^2}{x^2-1}$ .

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$

The second derivative is:

$$\begin{aligned} f''(x) &= \frac{-4(x^2-1)^2 - (-4x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} \\ &= \frac{-4(x^2-1) + 16x^2}{(x^2-1)^3} = \frac{12x^2 + 4}{(x^2-1)^3} \end{aligned}$$

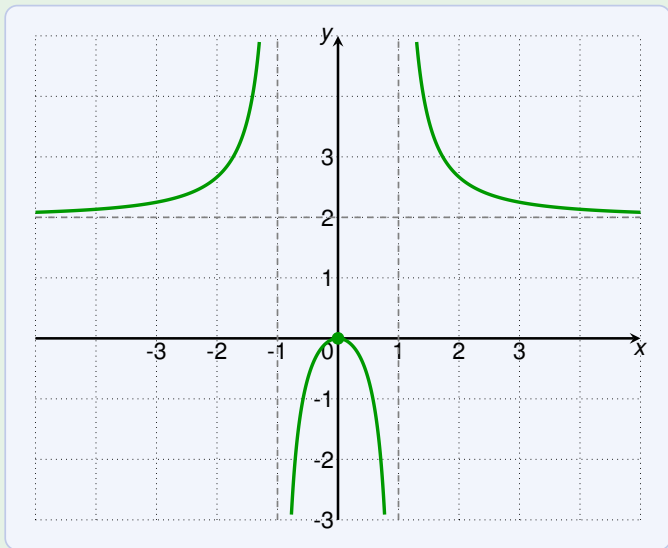
$12x^2 + 4 > 0$  for all  $x$

$$f''(x) > 0 \iff (x^2-1)^3 > 0 \iff x^2-1 > 0 \iff |x| > 1$$

- ▶ concave upward on  $(-\infty, -1) \cup (1, \infty)$
- ▶ concave downward on  $(-1, 1)$
- ▶ inflection points: none ( $-1$  and  $1$  not in the domain)

# Curve Sketching

Sketch the curve of  $f(x) = \frac{2x^2}{x^2-1}$ .



# Slant Asymptotes

Asymptotes that are neither horizontal nor vertical:

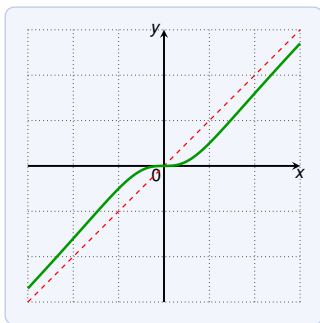
If

$$\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$$

or

$$\lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0$$

the the line  $y = mx + b$  is called **slant asymptote**.



Note that the distance between curve and line approaches 0.



# Slant Asymptotes

Sketch the graph of  $f(x) = \frac{x^3}{2x^2+1}$ .

The domain is  $(-\infty, \infty)$

The  $f(0) = 0$  and  $f(x) = 0 \iff x = 0$

Vertical asymptotes: none. Horizontal asymptotes: none

Slant asymptotes:  $y = \frac{1}{2}x$  since

$$\begin{aligned}\lim_{x \rightarrow \infty} \left( \frac{x^3}{2x^2+1} - \frac{x}{2} \right) &= \lim_{x \rightarrow \infty} \left( \frac{2x^3 - x(2x^2+1)}{2(2x^2+1)} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{-x}{2(2x^2+1)} \right) = 0\end{aligned}$$

# Slant Asymptotes

Sketch the graph of  $f(x) = \frac{x^3}{2x^2+1}$ .

$$f'(x) = \frac{3x^2(2x^2+1) - x^3(4x)}{(2x^2+1)^2} = \frac{2x^4+3x^2}{(2x^2+1)^2} = \frac{x^2(2x^2+3)}{(2x^2+1)^2}$$

Thus  $f'(x) > 0$  for all  $x \neq 0$ . Hence increasing on  $(-\infty, \infty)$ .

Local minima, maxima: none (since  $f'$  does not change sign)

We have

$$f''(x) = -\frac{2x(2x^2-3)}{(2x^2+1)^3}$$

Thus  $f''(x) = 0 \iff x = 0$  or  $x = \pm\sqrt{3/2}$

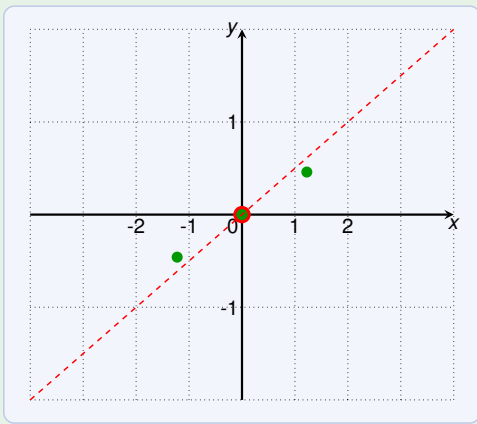
Interval	$f''(x)$	
$x < -\sqrt{3/2}$	+	concave up on $(-\infty, -\sqrt{3/2})$
$-\sqrt{3/2} < x < 0$	-	concave down on $(-\sqrt{3/2}, 0)$
$0 < x < \sqrt{3/2}$	+	concave up on $(0, \sqrt{3/2})$
$\sqrt{3/2} < x$	-	concave up down $(\sqrt{3/2}, \infty)$

Inflection points:  $(-\sqrt{\frac{3}{2}}, -\frac{3}{8}\sqrt{\frac{3}{2}})$ ,  $(0, 0)$  and  $(\sqrt{\frac{3}{2}}, \frac{3}{8}\sqrt{\frac{3}{2}})$

# Slant Asymptotes

Sketch the graph of  $f(x) = \frac{x^3}{2x^2+1}$ .

- ▶  $x$ - and  $y$ -intercept:  $(0, 0)$
- ▶ inflection points:  $(-\sqrt{\frac{3}{2}}, -\frac{3}{8}\sqrt{\frac{3}{2}})$ ,  $(0, 0)$  and  $(\sqrt{\frac{3}{2}}, \frac{3}{8}\sqrt{\frac{3}{2}})$
- ▶ slant asymptote:  $y = \frac{1}{2}x$



# Slant Asymptotes

Sketch the graph of  $f(x) = \frac{x^3}{2x^2+1}$ .

- ▶ increasing on  $(-\infty, \infty)$  and  $f'(0) = 0$
- ▶ concave up on  $(-\infty, -\sqrt{3/2})$  and  $(0, \sqrt{3/2})$
- ▶ concave down on  $(-\sqrt{3/2}, 0)$  and  $(\sqrt{3/2}, \infty)$

