Calculus M211

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A limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

where both

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

is called indeterminate form of type $\frac{0}{0}$.

Often cancellation of common factors helps:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)x}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \text{and} \qquad \lim_{x \to 1} \frac{\ln x}{x-1}$$

A limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

where both

is

$$\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty$$
called indeterminate form of type $\frac{\infty}{\infty}$.

Often helps to divide by highest power of x in the denominator:

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x\to\infty}\frac{\ln x}{x-1}$$

L'Hospital's Rule

Suppose *f* and *g* are differentiable and $g'(x) \neq 0$ near *a*, and

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is $-\infty$ or ∞ .

(near *a* = on an open interval containing *a* except possibly *a* itself)

Before applying L'Hospital's Rule it is important to verify that:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm \infty$$
 and $\lim_{x \to a} g(x) = \pm \infty$

Find

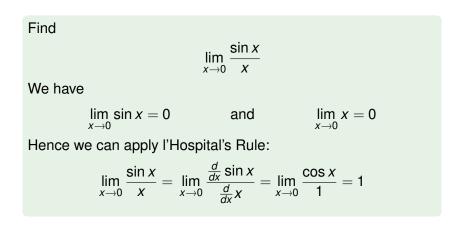
$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

We have

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 1} (x - 1) = 0$$

and hence we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} (x - 1)} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$



Find $\lim_{x\to\infty}\frac{e^x}{x^2}$ We have $\lim x^2 = \infty$ $\lim_{x\to\infty} e^x = \infty$ and $x \rightarrow \infty$ Hence we can apply l'Hospital's Rule: $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$ Again we have: lim $e^x = \infty$ and $\lim 2x = \infty$ $x \rightarrow \infty$ $x \rightarrow \infty$ So we can again use l'Hospital's Rule:

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{\frac{d}{dx}e^x}{\frac{d}{dx}2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

Find

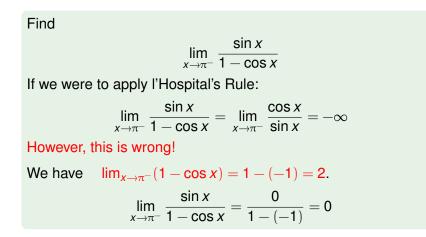
$$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}$$

We have

$$\lim_{x \to \infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x \to \infty} \sqrt[3]{x} = \infty$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \sqrt[3]{x}} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0$$



Before applying l'Hospital's Rule, always check that the limit is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's Rule is valid for one-sided limits and limits at infinity: $\lim_{x \to a^-} \frac{f(x)}{g(x)} \qquad \lim_{x \to a^+} \frac{f(x)}{g(x)} \qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} \qquad \lim_{x \to -\infty} \frac{f(x)}{g(x)}$

A limit of the form

 $\lim_{x\to a}(f(x)g(x))$

where

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty$$

is called indeterminate form of type $0 \cdot \infty$.

We then rewrite the limit as:

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} \frac{f(x)}{1/g(x)}$$

an indeterminate form of type $\frac{0}{0}$, or as

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

an indeterminate form of type $\frac{\infty}{\infty}.$

Evaluate the limit $\lim x \ln x$ $x \rightarrow 0^+$ We have $\lim_{x \to \infty} \ln x = -\infty$ $\lim_{x\to 0^+} x = 0$ and $x \rightarrow 0^+$ Thus we can choose for rewriting to: $\lim_{x\to 0^+}\frac{\ln x}{1/x}$ $\lim_{x\to 0^+}\frac{x}{1/\ln x}$ or We choose the 2nd since the derivatives are easier: $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$

A limit of the form

$$\lim_{x\to a}(f(x)-g(x))$$

where

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \infty$$

is called **indeterminate form of type** $\infty - \infty$.

We then rewrite the limit as a quotient.

Evaluate the limit

 $\lim_{x\to (\pi/2)^-}(\sec x-\tan x)$ $\lim_{x\to(\pi/2)^-} \sec x = \infty$ and $\lim_{x\to(\pi/2)^-} \tan x = \infty$ Then We use a common denominator: $\lim_{x \to (\pi/2)^{-}} (\sec x - \tan x) = \lim_{x \to (\pi/2)^{-}} \frac{1 - \sin x}{\cos x}$ Now $\lim_{x \to (\pi/2)^{-}} (1 - \sin x) = 0$ and $\lim_{x \to (\pi/2)^{-}} \cos x = 0$ Hence we can apply l'Hospital's Rule: $\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x}$

$$=\lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x}=0$$

A limit of the form

 $\lim_{x \to a} [f(x)]^{g(x)}$

is an indeterminate form

- of type 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
- of type ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$
- of type 1^{∞} if $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$

Each of these cases can be treated by writing the limit as:

$$\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} e^{\ln\left([f(x)]^{g(x)}\right)}$$
$$= \lim_{x \to a} e^{g(x)\ln f(x)} = e^{\lim_{x \to a} g(x)\ln f(x)}$$

Other types are **not** indeterminate forms: 0^{∞} , 1^{0} and ∞^{1} .

Evaluate the limit

$$\lim_{x\to 0^+} x^x$$

Then
$$\lim_{x\to 0^+} x = 0.$$

We write the limit as:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln x^x}$$
$$= e^{\lim_{x \to 0^+} (x \ln x)}$$
$$= e^0$$
$$= 1$$

Evaluate the limit $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$ Then $\lim_{x\to 0^+} (1 + \sin 4x) = 1$ and $\lim_{x\to 0^+} \cot x = \infty$ We write the limit as: $\lim_{x \to 0^+} (1 + \sin 4x)^{cotx} = \lim_{x \to 0^+} e^{\ln(1 + \sin 4x)^{cotx}}$ $= e^{\lim_{x\to 0^+} (\cot x \cdot \ln(1+\sin 4x))}$ $\lim_{x \to 0^+} \left(\cot x \cdot \ln(1 + \sin 4x) \right) = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$ Now $\lim_{x\to 0^+} \ln(1 + \sin 4x) = 0$ and $\lim_{x\to 0^+} \tan x = 0$ Hence we can apply l'Hospital's Rule: $\lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\frac{4\cos 4x}{1 + \sin 4x}}{(\sec x)^2} = \frac{\binom{4}{1}}{1} = 4$ Thus $\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} = e^4$