

# Calculus M211

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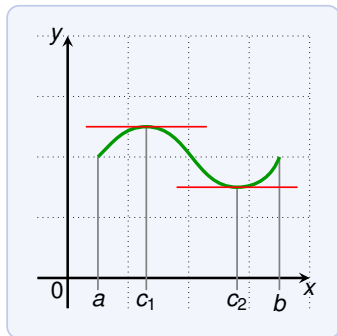
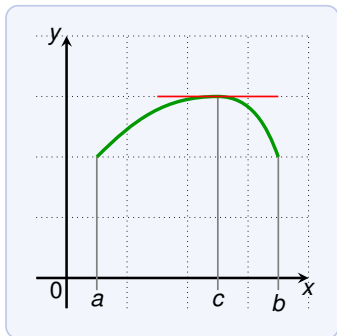
# Mean Value Theorem

## Rolle's Theorem

Let  $f$  be a function satisfying the all of the following:

- ▶  $f$  is continuous on  $[a, b]$
- ▶  $f$  is differentiable on  $(a, b)$
- ▶  $f(a) = f(b)$

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .



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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

## Proof.

- ▶ If  $f$  is constant, then  $f'(c) = 0$  for all  $c$  in  $(a, b)$ .
- ▶ If  $f$  is not constant, then there is  $x$  in  $(a, b)$  such that
$$f(x) > f(a) \qquad \text{or} \qquad f(x) < f(a)$$

Assume  $f(x) > f(a)$ . By the Extreme Value Theorem there is a  $c$  in  $[a, b]$  such that  $f(c)$  is the absolute maximum.

Then  $c$  must be in  $(a, b)$  and hence is a local maximum. Hence  $f'(c) = 0$  by Fermat's Theorem. □

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

Let  $s(t)$  be the position of an object after time  $t$ .

The object is in the same place at time  $t = 2s$  and  $t = 10s$ .

What does Rolle's Theorem tell us about the object?

It tells that there is a time  $c$  between  $2s$  and  $10s$  such that the

$$s'(t) = 0$$

that is, the velocity of the object at time  $c$  is 0.

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

Show that the function  $f$  is one-to-one (never takes the same value twice):

$$f(x) = x^3 + x - 1$$

Assume there would be  $x_1 < x_2$  such that  $f(x_1) = f(x_2)$ .

The function  $f$  is continuous and differentiable on  $[x_1, x_2]$ .

By Rolle's Theorem there exists  $c$  in  $(x_1, x_2)$  with  $f'(c) = 0$ .

This is a contradiction since  $f'(x) = 3x^2 + 1 \geq 1$  for all  $x$ .

There no  $x_1 < x_2$  such that  $f(x_1) = f(x_2)$ . Thus  $f$  is one-to-one.

# Mean Value Theorem

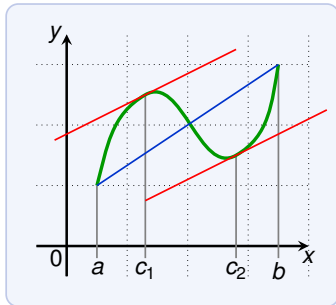
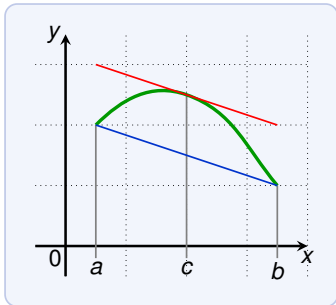
## Mean Value Theorem

Let  $f$  be a function satisfying the all of the following:

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- ▶  $f$  is differentiable on  $(a, b)$

Then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{this is the slope from } (a, f(a)) \text{ to } (b, f(b))$$

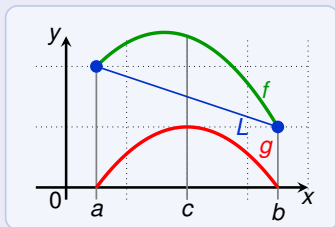


# Mean Value Theorem

## Proof of the Mean Value Theorem

Let  $f$  be a function satisfying the all of the following:

- ▶  $f$  is continuous on  $[a, b]$
- ▶  $f$  is differentiable on  $(a, b)$



Let  $L = mx + n$  be the line through  $(a, f(a))$  and  $(b, f(b))$ .

Define  $g = f - L$ . Then  $g(a) = 0$  and  $g(b) = 0$ .

By Rolle's Theorem there is  $c$  in  $(a, b)$  such that  $g'(c) = 0$ .

Since  $f = g + L$  we get  $f'(c) = g'(c) + m = m = \frac{f(b)-f(a)}{b-a}$ .

# Mean Value Theorem

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

Consider the function

$$f(x) = x^3 - x$$

on the interval  $[a, b]$  with  $a = 0$  and  $b = 2$ .

This is a polynomial, thus continuous and differentiable on  $[0, 2]$ .

By the Mean Value Theorem, there is a  $c$  in  $(0, 2)$  such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{6}{2} = 3$$

Indeed, we can find such a  $c$ , namely:  $f'\left(\frac{2}{\sqrt{3}}\right) = 3$ .



# Mean Value Theorem

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

Let  $s(t)$  be the position of an object after time  $t$ .

Then the average velocity between time  $t = a$  and  $t = b$  is:

$$\frac{s(b) - s(a)}{b - a}$$

What does the Mean Value Theorem tell us?

It states that there is a time  $c$  between  $a$  and  $b$  such that

$$f'(c) = \frac{s(b) - s(a)}{b - a}, \quad \text{that is}$$

the instantaneous velocity at  $c$  is equal to the average velocity.

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

We can interpret the Mean Value Theorem as follows:

There is a number  $c$  in the interval  $(a, b)$  such that the instantaneous rate of change at  $c$  is equal to the average rate of change over the interval  $[a, b]$ .

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

Suppose that  $f(0) = -3$  and  $f'(x) \leq 5$  for all  $x$ .

How large can  $f(2)$  possibly be?

By assumption,  $f$  is differentiable, and hence continuous.

By the Mean Value Theorem for the interval  $[0, 2]$ :

There exists  $c$  in  $(0, 2)$  such that  $f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) + 3}{2}$ .

We have:

$$5 \geq f'(c) = \frac{f(2) + 3}{2} \implies 10 \geq f(2) + 3 \implies 7 \geq f(2)$$

Thus the largest possible value for  $f(2)$  is 7.

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Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

Important consequences of the Mean Value Theorem are:

If  $f'(x) = 0$  for all  $x$  in  $(a, b)$  then  $f$  is constant on  $(a, b)$ .

(Proof like the previous example)

If  $f'(x) = g'(x)$  for all  $x$  in  $(a, b)$  then  $f - g$  is constant on  $(a, b)$ .

(In other words, then  $f(x) = g(x) + k$  for a constant  $k$ )

## Proof.

Let  $h = f - g$ . Then  $h' = f' - g' = 0$  on  $(a, b)$ . Thus  $h$  is constant on  $(a, b)$ . □