Calculus M211

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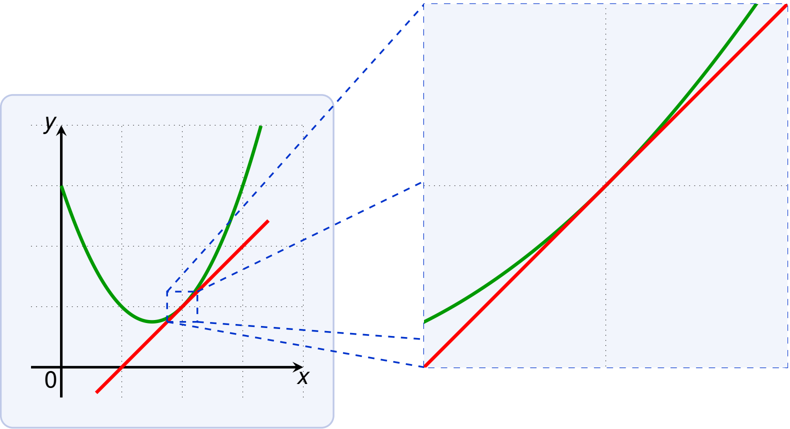
2013

Linear Approximation and Differentials



A curve is very close to its tangent close to the point of tangency (touching).

We can use this for approximating values of the function. . .

Linear Approximation and Differentials

Why approximate values of a function using a tangent?

- might be easy to compute $f(a)$ and $f'(a)$,
- but difficult to compute values $f(x)$ with x near a

We use the tangent line at $(a, f(a))$ to approximate $f(x)$ when x is close to a.

The tangent at $(a, f(a))$ is:

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

This function is called **linearization** of f at a.

When x is close to a, we approximate $f(x)$ by:

$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

This is called

- **linear approximation** of f at a, or
- **tangent line approximation** of f at a.

Linear Approximation and Differentials

Find the linearization of $f(x) = \sqrt{x+3}$ at 1 and use it to approximate $\sqrt{3.98}$.

We have:

$$f(1) = \sqrt{3+1} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+3}} \qquad\qquad f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

Thus the linearization of f at 1 is:

$$L(x) = 2 + \frac{1}{4}(x-1)$$

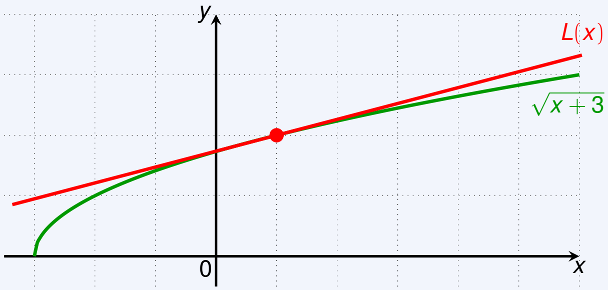
Thus for x close to 1 we approximate $f(x)$ by:

$$f(x) = \sqrt{x+3} \quad \approx \quad 2 + \frac{1}{4}(x-1)$$

In particular:

$$\sqrt{3.98} = \sqrt{0.98+3} \approx 2 + \frac{1}{4}(0.98-1) = 2 - 0.005 = 1.995$$

Linear Approximation and Differentials



The linear approximation is close to the curve when x is near 1.

Linear Approximation and Differentials

What is the linear approximation of $f(x) = \sin x$ at 0?
Use it to approximate $\sin 0.01$.

We have:

$$f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \qquad\qquad f'(0) = 1$$

Thus the linear approximation of $\sin x$ at 0 is:

$$L(x) = 0 + 1(x - 0) = x$$

We use this to approximate $\sin 0.01$:

$$\sin 0.01 \approx L(0.01) = 0.01$$

Linear Approximation and Differentials

What is the linear approximation of $f(x) = \cos x$ at 0?
Use it to approximate $\cos 0.01$.

We have:

$$f(0) = \cos 0 = 1$$

$$f'(x) = -\sin x \qquad\qquad f'(0) = 0$$

Thus the linear approximation of $\cos x$ at 0 is:

$$L(x) = 1 + 0(x - 0) = 1$$

We use this to approximate $\cos 0.01$:

$$\cos 0.01 \approx L(0.01) = 1$$

Approximations for sin and cos are often applied in physics (e.g. optics).

Linear Approximation and Differentials

Final Exam 2005

Use differential approximation, or the linearization method, to approximate $\sqrt[4]{15.5}$.

We have $f(x) = \sqrt[4]{x}$.

We need to choose where to compute the linearization: $a = 16$.

$$f(16) = 2$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} \qquad f'(16) = \frac{1}{4}16^{-\frac{3}{4}} = \frac{1}{4}\sqrt[4]{16}^{-3} = \frac{1}{4}\cdot\frac{1}{8} = \frac{1}{32}$$

The linearization of f at 16 is:

$$L(x) = 2 + \frac{1}{32}(x - 16)$$

Then the approximation of $\sqrt[4]{15.5}$ is:

$$\sqrt[4]{15.5} \approx L(15.5) = 2 + \frac{1}{32}(15.5 - 16) = 2 - \frac{1}{64} = \frac{127}{64}$$

Linear Approximation and Differentials

Final Exam 2004

Use the linearization method to approximate $(1.98)^4$.

We have $f(x) = x^4$.

We need to choose where to compute the linearization: $a = 2$.

$$f(2) = 16$$

$$f'(x) = 4x^3 \qquad f'(2) = 4 \cdot 2^3 = 4 \cdot 8 = 32$$

The linearization of f at 2 is:

$$L(x) = 16 + 32(x - 2)$$

Then the approximation of $(1.98)^4$ is:

$$(1.98)^4 \approx L(1.98) = 16 + 32(1.98 - 2) = 16 + 32(-0.02)$$

$$= 16 + 32\left(-\frac{1}{50}\right) = 16 - \frac{16}{25} = \frac{16 \cdot 24}{25}$$

Linear Approximation and Differentials

Final Exam 2003 (Spring)

Use differentials or the linearization approximation method to approximate $\ln(0.9)$.

We have $f(x) = \ln x$.

We need to choose where to compute the linearization: $a = 1$.

$$f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad\qquad f'(1) = 1$$

The linearization of f at 1 is:

$$L(x) = 0 + 1(x - 1) = x - 1$$

Then the approximation of $\ln(0.9)$ is:

$$\ln(0.9) \approx L(0.9) = 0.9 - 1 = -0.1$$

Linear Approximation and Differentials

Final Exam 2003 (Fall)

Use differentials to approximate $\sqrt[3]{999}$.

We have $f(x) = \sqrt[3]{x}$.

We choose where to compute the linearization: $a = 1000$.

$$f(1000) = 10$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2} \qquad f'(1000) = \frac{1}{3 \cdot 10^2} = \frac{1}{300}$$

The linearization of f at 1000 is:

$$L(x) = 10 + \frac{1}{300}(x - 1000)$$

Then the approximation of $\sqrt[3]{999}$ is:

$$\sqrt[3]{999} \approx L(999) = 10 + \frac{1}{300}(999 - 1000) = 10 - \frac{1}{300} = \frac{2999}{300}$$

Linear Approximation and Differentials

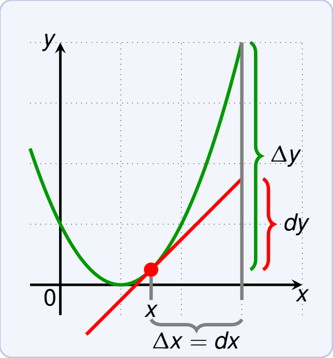
The method of linear approximation with differentials:

$$f'(x) = \frac{dy}{dx}$$

We view *dx* and *dy* as variables, then:

$$dy = f'(x)\ dx$$

So *dy* depends on the value of *x* and *dx*.



- x = point of linearization
- $\Delta x = dx$ is the distance from x
- dy = change of y of tangent
- Δy = change of y of curve f

As formulas:

- $dy = f'(x)\ dx$
- $\Delta y = f(x + \Delta x) - f(x)$