

# Calculus M211

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## 2nd Midterm Exam - Review

Find the derivative of  $f(x)$  when  $0 < x < 5$ :

$$f(x) = \frac{|x|}{\sqrt{5^2 - x^2}}$$

If  $0 < x$  then  $|x| = x$ . Then

$$\begin{aligned} f(x) &= \frac{x}{\sqrt{5^2 - x^2}} \\ f'(x) &= \frac{1 \cdot \sqrt{5^2 - x^2} - x \cdot \frac{1}{2}(5^2 - x^2)^{-\frac{1}{2}} \cdot (-2x)}{(\sqrt{5^2 - x^2})^2} \\ &= \frac{(5^2 - x^2) + x^2}{(\sqrt{5^2 - x^2})^3} = \frac{25}{(5^2 - x^2)^{\frac{3}{2}}} \end{aligned}$$

What is the left-hand derivative at 0? Then  $x < 0$ , thus  $|x| = -x$ .

left-hand derivative at 0 =  $-1/5$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = \left( \frac{1}{x^3} - 3\frac{1}{x} \right) \cdot (2x^4 + 5x)$$

using the product rule.

$$\begin{aligned} f'(x) &= \left( \frac{1}{x^3} - 3\frac{1}{x} \right) \cdot \frac{d}{dx}(2x^4 + 5x) + (2x^4 + 5x) \frac{d}{dx} \left( \frac{1}{x^3} - 3\frac{1}{x} \right) \\ &= \left( \frac{1}{x^3} - 3\frac{1}{x} \right) \cdot (8x^3 + 5) + (2x^4 + 5x) \left( -3x^{-4} - 3 \cdot (-1)x^{-2} \right) \\ &= \left( \frac{1}{x^3} - 3\frac{1}{x} \right) \cdot (8x^3 + 5) + (2x^4 + 5x) \left( \frac{-3}{x^4} + \frac{3}{x^2} \right) \\ &= \left( 8 - 24x^2 + \frac{5}{x^3} - \frac{15}{x} \right) + \left( -6 + 6x^2 - \frac{15}{x^3} + \frac{15}{x} \right) \\ &= 2 - 18x^2 - \frac{10}{x^3} \end{aligned}$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = \frac{\sqrt{x} - 2}{\sqrt{x} + 2}$$

using the quotient rule.

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x} + 2) \frac{d}{dx}(\sqrt{x} - 2) - (\sqrt{x} - 2) \frac{d}{dx}(\sqrt{x} + 2)}{(\sqrt{x} + 2)^2} \\ &= \frac{(\sqrt{x} + 2) \frac{1}{2\sqrt{x}} - (\sqrt{x} - 2) \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 2)^2} \\ &= \frac{(\sqrt{x} + 2) - (\sqrt{x} - 2)}{2\sqrt{x}(\sqrt{x} + 2)^2} \\ &= \frac{4}{2\sqrt{x}(\sqrt{x} + 2)^2} = \frac{2}{\sqrt{x}(\sqrt{x} + 2)^2} \end{aligned}$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = \frac{\sqrt{x^3} + 4}{3x^2 + 7}$$

using the quotient rule.

$$\begin{aligned} f'(x) &= \frac{(3x^2 + 7) \cdot \frac{d}{dx}(\sqrt{x^3} + 4) - (\sqrt{x^3} + 4) \cdot \frac{d}{dx}(3x^2 + 7)}{(3x^2 + 7)^2} \\ &= \frac{(3x^2 + 7) \cdot \frac{d}{dx}(x^{\frac{3}{2}} + 4) - (\sqrt{x^3} + 4) \cdot \frac{d}{dx}(3x^2 + 7)}{(3x^2 + 7)^2} \\ &= \frac{(3x^2 + 7) \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} - (\sqrt{x^3} + 4) \cdot 6x}{(3x^2 + 7)^2} \end{aligned}$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = 2x \cdot \sin(x) \cdot \cos(x)$$

We have:

$$f(x) = 2x \cdot (\sin(x) \cdot \cos(x))$$

$$\begin{aligned} f'(x) &= 2x \cdot \frac{d}{dx}(\sin(x) \cdot \cos(x)) + (\sin(x) \cdot \cos(x)) \cdot \frac{d}{dx}2x \\ &= 2x \cdot (\sin(x) \cdot (-\sin(x)) + \cos(x) \cdot \cos(x)) \\ &\quad + (\sin(x) \cdot \cos(x)) \cdot 2 \\ &= 2x \cdot (\cos^2(x) - \sin^2(x)) + 2 \sin(x) \cos(x) \\ &= 2x \cdot \cos(2x) + \sin(2x) \end{aligned}$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = 8x^2 + 2e^x$$

We have:

$$f'(x) = 16x + 2e^x$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = x \cdot e^{\frac{1}{x}}$$

We have:

$$\begin{aligned} f'(x) &= x \cdot \frac{d}{dx} e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{d}{dx} x \\ &= x \cdot e^{\frac{1}{x}} \cdot \frac{d}{dx} \frac{1}{x} + e^{\frac{1}{x}} \\ &= x \cdot e^{\frac{1}{x}} \cdot (-1)x^{-2} + e^{\frac{1}{x}} \\ &= e^{\frac{1}{x}} \left( 1 - \frac{1}{x} \right) \end{aligned}$$



## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = \frac{8e^x}{e^x + 1}$$

We have:

$$\begin{aligned} f'(x) &= \frac{(e^x + 1) \cdot \frac{d}{dx}(8e^x) - 8e^x \cdot \frac{d}{dx}(e^x + 1)}{(e^x + 1)^2} \\ &= \frac{(e^x + 1) \cdot 8e^x - 8e^x \cdot e^x}{(e^x + 1)^2} \\ &= \frac{8e^x}{(e^x + 1)^2} \end{aligned}$$

## 2nd Midterm Exam - Review

Find the derivative of

$$f(x) = \cos(\sin(x^2))$$

We have:

$$\begin{aligned} f'(x) &= -\sin(\sin(x^2)) \cdot \frac{d}{dx}(\sin(x^2)) \\ &= -\sin(\sin(x^2)) \cdot \cos(x^2) \cdot \frac{d}{dx}x^2 \\ &= -\sin(\sin(x^2)) \cdot \cos(x^2) \cdot 2x \end{aligned}$$

## 2nd Midterm Exam - Review

Use logarithmic differentiation to find the derivative of

$$y = \sqrt{\frac{x-1}{x^4+1}}$$

We have:

$$\ln y = \ln \sqrt{\frac{x-1}{x^4+1}} = \frac{1}{2} \cdot \ln \frac{x-1}{x^4+1} = \frac{1}{2} \cdot (\ln(x-1) - \ln(x^4+1))$$

Thus

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[ \frac{1}{2} \cdot (\ln(x-1) - \ln(x^4+1)) \right]$$

$$\frac{1}{y} y' = \frac{1}{2} \cdot \left( \frac{d}{dx} \ln(x-1) - \frac{d}{dx} \ln(x^4+1) \right)$$

$$y' = \frac{1}{2} y \cdot \left( \frac{1}{x-1} - \frac{1}{x^4+1} 4x^3 \right)$$

$$y' = \frac{1}{2} \cdot \sqrt{\frac{x-1}{x^4+1}} \cdot \left( \frac{1}{x-1} - \frac{1}{x^4+1} 4x^3 \right)$$

## 2nd Midterm Exam - Review

Find the derivative of

$$y = \sqrt{x} \cdot (1 + x^2)^{\sin(x)}$$

We have:

$$\begin{aligned}\ln y &= \ln \left[ \sqrt{x} \cdot (1 + x^2)^{\sin(x)} \right] = \ln \sqrt{x} + \ln(1 + x^2)^{\sin(x)} \\ &= \frac{1}{2} \cdot \ln x + \sin(x) \cdot \ln(1 + x^2)\end{aligned}$$

Thus

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[ \frac{1}{2} \cdot \ln x + \sin(x) \cdot \ln(1 + x^2) \right]$$

$$\frac{y'}{y} = \left[ \frac{1}{2x} + \sin(x) \cdot \frac{d}{dx} \ln(1 + x^2) + \ln(1 + x^2) \cdot \frac{d}{dx} \sin(x) \right]$$

$$y' = y \left[ \frac{1}{2x} + \sin(x) \frac{1}{1 + x^2} 2x + (\ln(1 + x^2)) \cdot \cos(x) \right]$$

$$y' = \left( \sqrt{x} \cdot (1 + x^2)^{\sin(x)} \right) \left[ \frac{1}{2x} + \frac{2x \sin(x)}{1 + x^2} + \cos(x) \ln(1 + x^2) \right]$$

## 2nd Midterm Exam - Review

Let  $f(x) = x^2$ . Find  $a > 0$  such that the tangent to the curve at the point  $(a, f(a))$  passes through the point  $(1, -3)$ .

We have  $f'(x) = 2x$ . Thus the tangent through  $(a, f(a))$  is

$$y - f(a) = f'(a) \cdot (x - a) \quad \implies \quad y - a^2 = 2a(x - a)$$

We want that the tangent passes through  $(1, -3)$ , thus

$$\begin{aligned}(-3) - a^2 &= 2a(1 - a) \\ \implies (-3) - a^2 &= 2a - 2a^2 \\ \implies a^2 - 2a - 3 &= 0 \\ \implies a &= 1 \pm \sqrt{1 + 3}\end{aligned}$$

Thus  $a = 3$  (recall that we were searching for  $a > 0$ )

The tangent is  $y - 9 = 6(x - 3)$

## 2nd Midterm Exam - Review

Show that  $f(x) = 2e^x + 3x + 15x^3$  has no tangent with slope 2.

We have:

$$f'(x) = 2e^x + 3 + 5x^2$$

Note that

$$e^x \geq 0 \quad \text{for all } x$$

$$x^2 \geq 0 \quad \text{for all } x$$

and thus

$$f'(x) = 2e^x + 3 + 5x^2 \geq 3$$

The slope of the curve  $f(x)$  is  $\geq 3$  everywhere.  
Hence the curve cannot have a tangent with slope 2.

## 2nd Midterm Exam - Review

Where does the normal line to  $f(x) = x - x^2$  at point  $(1, 0)$  intersect the the curve the second time?

We have:

$$f'(x) = 1 - 2x \quad \implies \quad f'(1) = -1$$

The normal line at  $(1, 0)$  has slope  $-\frac{1}{-1} = 1$ , and thus is

$$y - 0 = 1 \cdot (x - 1) \qquad y = x - 1$$

We look for the intersection of  $f(x)$  and the normal line:

$$\begin{aligned} x - 1 &= x - x^2 & \implies & \quad x^2 - 1 = 0 \\ & & \implies & \quad (x - 1) \cdot (x + 1) = 0 \end{aligned}$$

Thus the second intersection is at point  $(-1, -2)$ .

## 2nd Midterm Exam - Review

Find constants  $A, B, C$  such that  $y = Ax^2 + Bx + C$  satisfies

$$y'' + y' - 2y = x^2$$

We have

$$y = Ax^2 + Bx + C \quad y' = 2Ax + B \quad y'' = 2A$$

Thus

$$\begin{aligned} x^2 &= y'' + y' - 2y = (2Ax + B) + (2A) - 2(Ax^2 + Bx + C) \\ &= (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) \end{aligned}$$

Hence

$$-2A = 1 \implies A = -1/2$$

$$2A - 2B = 0 \implies B = A = -1/2$$

$$2A + B - 2C = 0 \implies C = -3/4$$



## 2nd Midterm Exam - Review

Let  $c > \frac{1}{2}$ . How many lines through the point  $(0, c)$  are normal lines to  $f(x) = x^2$ ?

Let  $a$  be arbitrary. We construct the normal line at  $(a, f(a))$ :

$$f'(a) = 2a$$

As a consequence the normal line at  $(a, f(a))$  for  $a \neq 0$  is:

$$y - a^2 = -\frac{1}{2a}(x - a)$$

We check for which  $a$  the normal goes through  $(0, c)$ :

$$c - a^2 = -\frac{1}{2a}(0 - a) \quad \implies \quad a^2 = c - \frac{1}{2}$$

Note that  $c - \frac{1}{2} > 0$ ! Thus there are two solutions for  $a$ .

Note that the normal at  $(0, 0)$  is vertical and goes through  $(0, c)$ !

Hence there are three normal lines going through  $(0, c)$ .

## 2nd Midterm Exam - Review

Is there a line that is tangent to both curves  $f$  and  $g$ ?

$$f(x) = x^2 \qquad g(x) = x^2 - 2x + 2$$

We compute the tangent to  $f$  at  $(a, f(a))$ :

$$\begin{aligned} f'(a) = 2a &\implies y - a^2 = 2a(x - a) \\ &\implies y = (2a)x + (-a^2) \end{aligned}$$

We compute the tangent to  $g$  at  $(b, g(b))$ :

$$\begin{aligned} g'(b) = 2b - 2 &\implies y - (b^2 - 2b + 2) = (2b - 2)(x - b) \\ &\implies y = (2b - 2)x + (-b^2 + 2) \end{aligned}$$

The tangents are equal if:

$$\begin{aligned} 2a = 2b - 2 &\implies a = b - 1 \implies a = 1/2 \\ -a^2 = -b^2 + 2 &\implies -(b - 1)^2 = -b^2 + 2 \\ &\implies -(b^2 - 2b + 1) = -b^2 + 2 \implies b = 3/2 \end{aligned}$$

Thus the line  $y = x - 1/4$  is tangent to both curves.

## 2nd Midterm Exam - Review

Find all points on the curve where the slope of the tangent is  $-1$

$$x^2y^2 + xy = 2$$

We use implicit derivatives:

$$\frac{d}{dx}(x^2y^2 + xy) = \frac{d}{dx}2$$

$$\implies x^2 \frac{d}{dx}y^2 + y^2 \frac{d}{dx}x^2 + x \frac{d}{dx}y + y \frac{d}{dx}x = 0$$

$$\implies x^2 2yy' + y^2 2x + xy' + y = 0$$

$$\stackrel{y'=-1}{\implies} -2x^2y + 2xy^2 - x + y = 0$$

$$\implies (-x + y) \cdot (2xy + 1) = 0$$

## 2nd Midterm Exam - Review

Find all points on the curve where the slope of the tangent is  $-1$

$$x^2y^2 + xy = 2$$

The slope is  $-1$  if  $x = y$  or  $xy = -1/2$  .

But when is  $(x, y)$  on the original curve?

$$x = y \implies (x^2)^2 + x^2 = 2$$

$$\implies (x^2 - 1)(x^2 + 2) = 0$$

$$\implies x^2 = 1$$

$$\implies x = \pm 1$$

$$\implies \text{on the curve if } x = y = \pm 1$$

$$xy = -1/2 \implies (xy)^2 + xy = 2$$

$$\implies 1/4 + 1/2 = 2$$

$$\implies \text{can never be on the curve}$$

The points on the curve with slope  $-1$  are  $(1, 1)$  and  $(-1, -1)$ .

## 2nd Midterm Exam - Review

Find an equation for the tangent at point  $(3, -2)$  to the curve

$$y^2(y^2 - 4) = x^2(x^2 - 9)$$

We have

$$y^4 - 4y^2 = x^4 - 9x^2$$

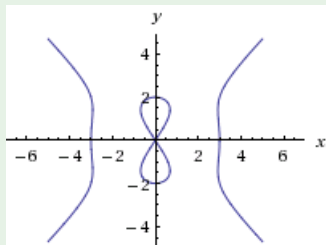
We use implicit differentiation

$$\frac{d}{dx}(y^4 - 4y^2) = \frac{d}{dx}(x^4 - 9x^2)$$

$$\implies 4y^3y' - 8yy' = 4x^3 - 18x$$

$$\implies y'(4y^3 - 8y) = 4x^3 - 18x$$

$$\implies y' = \frac{2x^3 - 9x}{2y^3 - 4y} = \frac{2 \cdot 3^3 - 9 \cdot 3}{2(-2)^3 - 4 \cdot (-2)} = \frac{54 - 27}{-16 + 8} = -\frac{27}{8}$$



Thus the equation for the tangent is  $y + 2 = -\frac{27}{8} \cdot (x - 3)$ .

## 2nd Midterm Exam - Review

Evaluate the limit

$$\lim_{x \rightarrow \infty} \left( x \cdot \sin\left(\frac{1}{x}\right) \right)$$

We have

$$\lim_{x \rightarrow \infty} \left( x \cdot \sin\left(\frac{1}{x}\right) \right) \stackrel{\text{take } h = \frac{1}{x}}{=} \lim_{h \rightarrow 0} \left( \frac{1}{h} \cdot \sin(h) \right) = 1$$

## 2nd Midterm Exam - Review

Find the derivative of

$$\begin{aligned}f(x) &= (\sqrt{x^5} - 3\sqrt[3]{x}) \cdot (6x^4 + 2x) \\ &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot (6x^4 + 2x)\end{aligned}$$

We have:

$$\begin{aligned}f'(x) &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot \frac{d}{dx}(6x^4 + 2x) + (6x^4 + 2x) \cdot \frac{d}{dx}(x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \\ &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot (24x^3 + 2) + (6x^4 + 2x) \cdot \left(\frac{5}{2}x^{\frac{3}{2}} - x^{-\frac{2}{3}}\right)\end{aligned}$$