# Calculus M211

Jörg Endrullis

Indiana University Bloomington

2013

# Related (Dependent) Rates

Air is pumped into a spherical balloon:

- the volume increases with 100 $\text{cm}^3/\text{s}$

Find: rate of change of the radius when the diameter is $50cm$.

First step: introduce suggestive notation

- let $V(t)$ be the volume after time $t$
- let $r(t)$ be the radius after time $t$

Then the given problem translates to

$$V'(t) = 100\ \text{cm}^3/\text{s} \qquad \text{Find } r'(t) \text{ when } r = 25\text{cm}.$$

How are the volume of a sphere and its radius related?

$$V = \frac{4}{3}\pi r^3 \quad \text{thus} \quad V'(t) = \frac{d}{dt}\left(\frac{4}{3}\pi r(t)^3\right) = \frac{4}{3}\pi \cdot 3r(t)^2 r'(t)$$

We solve for $r'(t)$:

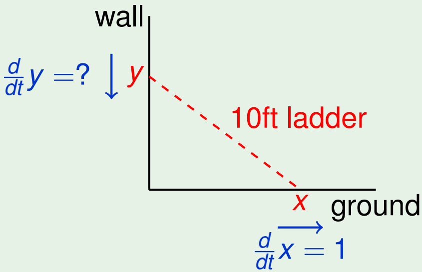
$$r'(t) = \frac{V'(t)}{4\pi \cdot r(t)^2} \qquad r'(t) = \frac{100}{4\pi \cdot 25^2} = \frac{1}{25\pi}\ \text{cm/s}$$

# Related (Dependent) Rates

A ladder of length 10ft rests against a vertical wall.

- the bottom of the ladder slides away from the wall with 1ft/s

How fast is the top sliding when the bottom is 6ft from the wall?




Thus

$$x^2 + y^2 = 10^2$$
$$\overset{x=6}{\Longrightarrow} 6^2 + y^2 = 10^2$$
$$\Longrightarrow y = \pm\sqrt{10^2 - 6^2}$$
$$\Longrightarrow y = 8$$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}10^2 \Longrightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$\Longrightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Longrightarrow \frac{dy}{dt} = -\frac{6}{8}\cdot 1 = -\frac{3}{4}$$

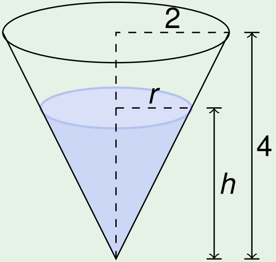
The top slides with $\frac{3}{4}$ft/s when the bottom is 6ft from the wall.

# Related (Dependent) Rates

A water tank has the shape of an inverted circular cone:

- base radius $2m$ and the height is $4m$,
- water is pumped into the tank at a rate of $2\text{m}^3/\text{min}$.

At what rate is the water rising when the water is 3m deep?




$$V = \frac{1}{3}\pi r^2 h$$

How is $r$ related to $h$?

$$\frac{r}{h} = \frac{2}{4} \implies r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi(\frac{1}{2}h)^2 h = \frac{1}{12}\pi h^3$$

We differentiate both sides with respect to $t$:

$$\frac{dV}{dt} = \frac{d}{dt}(\frac{1}{12}\pi h^3) = \frac{1}{12}\pi 3h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \overset{h=3}{=} \frac{4}{\pi 9} \cdot 2$$

Thus the water rises with $8/(\pi 9)$m/min when its is 3m deep.

# Related (Dependent) Rates

## Problem Solving Strategy

Important when solving textual problems:

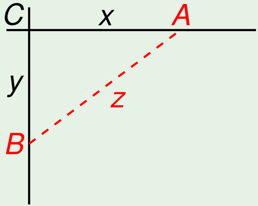
- Read the problem carefully.
- Draw a diagram.
- Introduce notation, function names for the quantities.
- Express given information and goal using the notation.
- Write equations relating the quantities. Eliminate dependent variables (in the previous example we have eliminated the radius as it was dependent on the height).
- Use the chain rule to differentiate both sides w.r.t. $t$.
- Solve for the unknown rate, and substitute the given information into the resulting formula.

# Related (Dependent) Rates

Two cars are headed for the same road intersection:

- car $A$ is traveling west with 50mi/h
- car $B$ is traveling north with 60mi/h

At what rate are the cars approaching
when $A$ is 0.3mi and $B$ is 0.4mi from the intersection?




- $x(t)$ = distance of $A$ to crossing
- $y(t)$ = distance of $B$ to crossing
- $z(t)$ = distance of $A$ to $B$

$$\frac{d}{dt}x = -50 \qquad \frac{d}{dt}y = -60$$

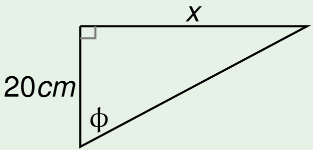
$$z^2 = x^2 + y^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt} \implies \frac{dz}{dt} = \frac{0.3}{0.5}(-50) + \frac{0.4}{0.5}(-60) = -78$$

When $x = 0.3$ & $y = 0.4$, we get $z = 0.5$. The answer is 78mi/h.

# Related (Dependent) Rates

We have a right-angled triangle of the form




The length $x$ increases with $4cm/s$.
How fast is the angle $\phi$ changing when $x = 15cm$?

The quantities $x$ and $\phi$ are related by:

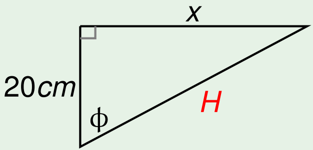
$$\tan\phi = \frac{x}{20}$$

Differentiating both sides yields:

$$\frac{d}{dt}\tan\phi = \frac{d}{dt}\frac{x}{20} \quad\Longrightarrow\quad \frac{1}{(\cos\phi)^2}\cdot\frac{d\phi}{dt} = \frac{1}{20}\cdot\frac{dx}{dt}$$

# Related (Dependent) Rates

We have a right-angled triangle of the form




The length $x$ increases with $4cm/s$.
How fast is the angle $\phi$ changing when $x = 15cm$?

$$\frac{1}{(\cos\phi)^2} \cdot \frac{d\phi}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

$$\Longrightarrow \quad \frac{d\phi}{dt} = \frac{(\cos\phi)^2}{20} \cdot \frac{dx}{dt} = \frac{(\cos\phi)^2}{20} \cdot 4 = \frac{(\cos\phi)^2}{5}$$

We have $\cos\phi = 20/H = 20/\sqrt{15^2 + 20^2} = 20/25 = 4/5$.
Thus

$$\frac{d\phi}{dt} = \left(\frac{4}{5}\right)^2 \cdot \frac{1}{5} = \frac{4^2}{5^3} = \frac{16}{125}\text{rad/s}$$