Calculus M211

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Derivatives of Logarithmic Functions

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Using implicit differentiation we get:

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Differentiate

$$y = \ln(x^3 + 1)$$

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$$y = \sqrt{\ln x}$$

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$$f(x) = \begin{cases} \ln x & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases}$$

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$$\frac{d}{dx} \ln |x| = \frac{1}{x} \quad \text{for all } x \neq 0$$

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$$\begin{aligned} y' &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{d}{dx} \frac{x+1}{\sqrt{x-2}} \\ &= \frac{\sqrt{x-2}}{x+1} \cdot \frac{1 \cdot \sqrt{x-2} - (x+1) \cdot \frac{d}{dx}\sqrt{x-2}}{(\sqrt{x-2})^2} \end{aligned}$$

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$$\begin{aligned}
y' &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{d}{dx}\frac{x+1}{\sqrt{x-2}} \\
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&= \frac{\sqrt{x-2}}{x+1} \cdot \frac{\sqrt{x-2} - (x+1) \cdot \frac{1}{2\sqrt{x-2}} \cdot 1}{(\sqrt{x-2})^2} \\
&= \frac{x-2-(x+1)\cdot\frac{1}{2}}{(x+1)(x-2)}
\end{aligned}$$

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Thus

$$y' = \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2}$$

Logarithmic Differentiation

The following method is called **logarithmic differentiation.**

Differentiate

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We use implicit differentiation:

$$\frac{d}{dx} \ln y = \frac{3}{4} \frac{d}{dx} \ln x + \frac{1}{2} \frac{d}{dx} \ln(x^2 + 1) - 5 \frac{d}{dx} \ln(3x + 2)$$

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$$\frac{1}{y}y' = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\frac{1}{3x+2}\cdot 3$$

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Hence

$$y' = \frac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5} \cdot \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

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- Use laws of logarithms to simplify.
- Differentiate implicitly with respect to x.
- Solve the resulting equation for y'.

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$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = y\left(\frac{2 + \ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

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Alternative: $x^{\sqrt{x}}$

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Alternative: $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$

Logarithmic Differentiation

Differentiate

$$y = x^{\sqrt{x}}$$

The power rule does not apply: the exponent contains x!

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$$e \quad = \quad \lim_{h\to 0} (1+h)^{\frac{1}{h}} \quad = \quad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$