Calculus M211

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Indiana University Bloomington

2013

Implicit Differentiation

Consider the equation:

$$x^2 + y^2 = 25$$

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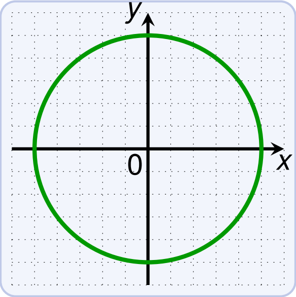
This equation describes a circle:



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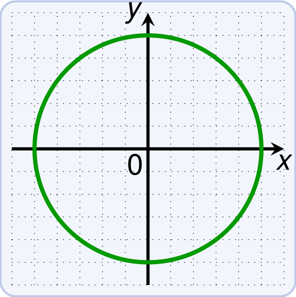

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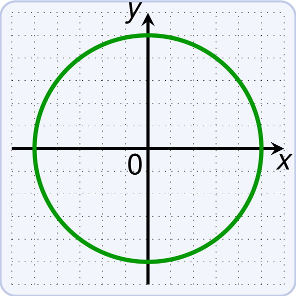

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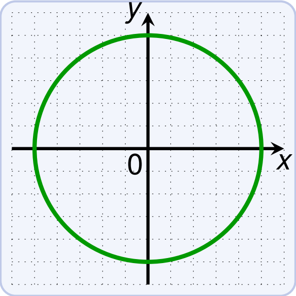

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How to compute the slope of points on this curve?

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Find an equation of the tangent at point $(3, 4)$.

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Find an equation of the tangent at point $(3, 4)$.

At point $(3, 4)$ we have:

$$\frac{dy}{dx} = -\frac{3}{4}$$

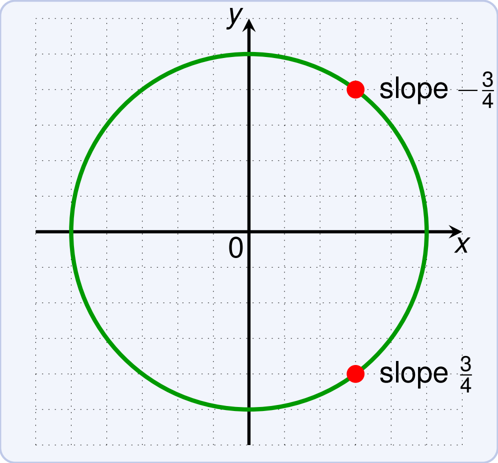
Thus the tangent is

$$y - 4 = -\frac{3}{4}(x - 3)$$

Implicit Differentiation

Note that the derivative now depends on x and y!

$$\frac{dy}{dx} = -\frac{x}{y}$$



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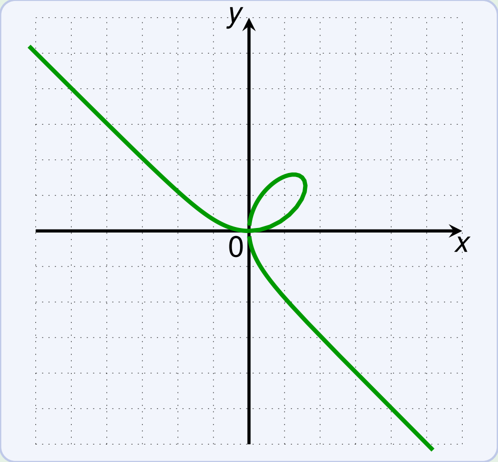
Find y' where $x^3 + y^3 = 6xy$.



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$$3y^2 \cdot y' - 6xy' = +6y - 3x^2$$

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$$y'(3y^2 - 6x) = 6y - 3x^2$$

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Thus the tangent is $\quad y - 3 = -1(x - 3)$.

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Since $x > 0$, we get $x = \sqrt[3]{16}$.

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Since $x > 0$, we get $x = \sqrt[3]{16}$. Then the point is

$$(\sqrt[3]{16}, \sqrt[3]{32})$$

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We have:

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y' &= -\frac{y^2\sin x + \cos(x+y)}{\cos(x+y) - 2y\cos x}
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$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

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Thus

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\frac{d}{dx}x^3 - x^3\frac{d}{dx}y^3}{(y^3)^2} = -\frac{y^3 3x^2 - x^3 3y^2 y'}{y^6}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} = -\frac{3x^2(x^4+y^4)}{y^7}$$

Implicit Differentiation

Find y'' where

$$x^4 + y^4 = 16$$

We have:

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}16 \quad\Longrightarrow\quad 4x^3 + 4y^3y' = 0 \quad\Longrightarrow\quad y' = -\frac{x^3}{y^3}$$

Thus

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\frac{d}{dx}x^3 - x^3\frac{d}{dx}y^3}{(y^3)^2} = -\frac{y^3 3x^2 - x^3 3y^2 y'}{y^6}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} = -\frac{3x^2(x^4 + y^4)}{y^7} = -\frac{48x^2}{y^7}$$