

Calculus M211

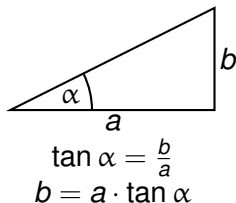
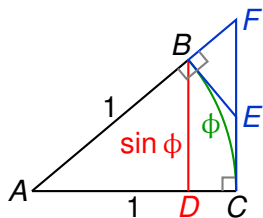
Jörg Endrullis

Indiana University Bloomington

2013

Derivatives of Trigonometric Functions

We investigate $\lim_{\phi \rightarrow 0} \frac{\sin \phi}{\phi}$ (for simplicity assume $0 < \phi < \pi/2$)



$$\sin \phi = |BD| < \phi \quad \Rightarrow \quad \frac{\sin \phi}{\phi} < 1$$

$$\phi < |CE| + |EB| \quad \& \quad |EB| < |EF| \quad \Rightarrow \quad \phi < |CE| + |EF| = |CF|$$

$$\phi < |CF| = 1 \cdot \tan \phi = \frac{\sin \phi}{\cos \phi} \quad \Rightarrow \quad \cos \phi < \frac{\sin \phi}{\phi} < 1$$

We use the Squeeze Theorem:

$$\lim_{\phi \rightarrow 0} \cos \phi = 1 = \lim_{\phi \rightarrow 0} 1 \quad \Rightarrow \quad \lim_{\phi \rightarrow 0} \frac{\sin \phi}{\phi} = 1$$

Derivatives of Trigonometric Functions

We have the following identities for sin and cos:

$$\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

Derivatives of Trigonometric Functions

We will prove that

$$\frac{d}{dx} \sin(x) = \cos(x)$$

We have

$$\begin{aligned} \frac{d}{dx} \sin(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{\sin x \cos h - \sin(x)}{h} + \frac{\cos x \sin h}{h} \right] \\ &= \lim_{h \rightarrow 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] \\ &= \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_1 \end{aligned}$$

Derivatives of Trigonometric Functions

We will prove that

$$\frac{d}{dx} \sin(x) = \cos(x)$$

We have

$$\begin{aligned} \frac{d}{dx} \sin(x) &= \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} \right) + \cos x \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{(\cos h)^2 - 1}{h(\cos h + 1)} \right) + \cos x \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{-(\sin h)^2}{h(\cos h + 1)} \right) + \cos x \\ &= \sin x \cdot \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \frac{-\sin h}{\cos h + 1} \right) + \cos x \\ &= \sin x \cdot 1 \cdot 0 + \cos x = \cos x \end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Differentiate

$$f(x) = x^2 \sin x$$

We have

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^2 \quad \text{product rule} \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Differentiate $\tan x$:

$$\begin{aligned}\frac{d}{dx} \tan x &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\ &= \frac{\cos x \cdot \frac{d}{dx} \sin x - \sin x \cdot \frac{d}{dx} \cos x}{(\cos x)^2} \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Differentiate the **secant** $\sec x = \frac{1}{\cos x}$:

$$\begin{aligned}\frac{d}{dx} \sec x &= \frac{d}{dx} \left(\frac{1}{\cos x} \right) \\ &= \frac{\cos x \cdot \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx} \cos x}{(\cos x)^2} \\ &= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x\end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Differentiate the **cosecant** $\csc x = \frac{1}{\sin x}$:

$$\begin{aligned}\frac{d}{dx} \csc x &= \frac{d}{dx} \left(\frac{1}{\sin x} \right) \\ &= \frac{\sin x \cdot \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx} \sin x}{(\sin x)^2} \\ &= \frac{-\cos x}{\sin^2 x} = -\csc x \cdot \cot x\end{aligned}$$

Derivatives of Trigonometric Functions

Summary: derivatives of trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cdot \cot x$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = \sin(x^2)$.

We have $f = g \circ h$ where $g(x) = \sin x$ and $h(x) = x^2$:

$$g'(x) = \cos x$$

$$h'(x) = 2x$$

$$f'(x) = g'(h(x)) \cdot h'(x) = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

Differentiate $g(x) = \sin^2 x = (\sin x)^2$.

We have $f = g \circ h$ where $g(x) = x^2$ and $h(x) = \sin x$:

$$g'(x) = 2x$$

$$h'(x) = \cos x$$

$$f'(x) = g'(h(x)) \cdot h'(x) = 2 \sin x \cdot \cos x$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = e^{\sin x}$.

We have $f = g \circ h$ where $g(x) = e^x$ and $h(x) = \sin x$:

$$g'(x) = e^x$$

$$h'(x) = \cos x$$

$$f'(x) = g'(h(x)) \cdot h'(x) = e^{\sin x} \cdot \cos x$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = \sin(\cos(\tan x))$.

$$\begin{aligned}f'(x) &= \cos(\cos(\tan x)) \cdot \frac{d}{dx} \cos(\tan x) \\&= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot \frac{d}{dx} \tan x \\&= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \frac{1}{\cos^2 x}\end{aligned}$$

Note that we have applied the chain rule twice!