Calculus M211

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Derivative as a Function

The **derivative of** f is a function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$.
- Geometrically, $f'(x)$ is the slope of the tangent at $(x, f(x))$.

Let $f(x) = x^3 - x$. Find a formula for $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) \\ &= 3x^2 - 1 \end{aligned}$$

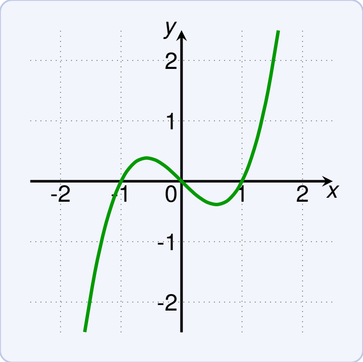
Exam Task from 2005

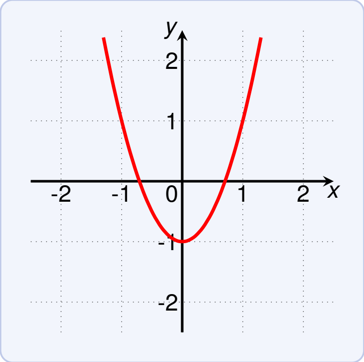
Using the definition of derivative, find $f'(x)$, where $f(x) = \sqrt{2x}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h}-\sqrt{2x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{h} \cdot \frac{\sqrt{2x+2h}+\sqrt{2x}}{\sqrt{2x+2h}+\sqrt{2x}} \right) \\
&= \lim_{h\to 0} \left(\frac{2x+2h-2x}{h\cdot(\sqrt{2x+2h}+\sqrt{2x})} \right) \\
&= \lim_{h\to 0} \left(\frac{2}{\sqrt{2x+2h}+\sqrt{2x}} \right) \\
&= \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}
\end{aligned}
$$

Derivative as a Function

Which of these functions is the derivative of the other?

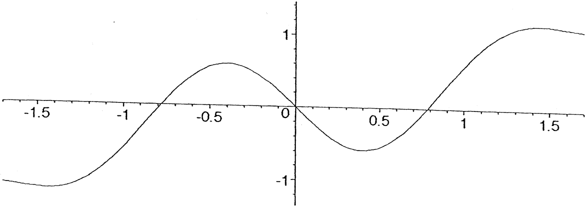




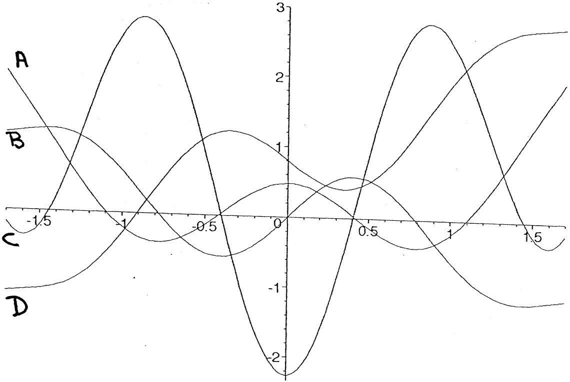
The right is the derivative of the left:

- look at local maxima and minima of f; then f' must be 0
- where f increases, f' must be positive
- where f decreases, f' must be negative

Exam Task from 2005



Which of the four labelled curves below best represents the graph of $y = f'(x)$?



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Derivative as a Function

Practice computing derivatives!

Try examples from the book for yourself. Among others:

- Example 3 in Section 2.8
- Example 4 in Section 2.8

Derivative as a Function

A function f is **differentiable at** a if $f'(a)$ exists.

A function f is **differentiable on an open interval** (a, b) if it is differentiable at every number of the interval.

Note that the interval (a, b) may be $(-\infty, b)$, (a, ∞) or $(-\infty, \infty)$.

Derivative as a Function

Where is $f(x) = |x|$ differentiable?

For $x > 0$ we have:

- $|x| = x$,
- $|x + h| = x + h$ for small enough h.

Thus for $x > 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} 1 = 1$$

For $x < 0$ we have:

- $|x| = -x$,
- $|x + h| = -x - h$ for small enough h.

Thus for $x < 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} -1 = -1$$

Derivative as a Function

Where is $f(x) = |x|$ differentiable?

For $x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

We need to look at the left and right limits:

$$\lim_{h \to 0^-} \frac{|h|}{h} \overset{\text{since } h<0}{=} \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

and

$$\lim_{h \to 0^+} \frac{|h|}{h} \overset{\text{since } h>0}{=} \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

The left and right limits are different.

Thus $f'(0)$ does not exist, and $f(x)$ is not differentiable at 0.

Hence f is differentiable at all numbers in $(-\infty, 0) \cup (0, \infty)$.

Derivatives and Continuity

If f is differentiable at a, then f is continuous at a.

The proof is in the book. Intuitively it holds because...

Differentiable at a means:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists}$$

Continuous at a means:

$$\lim_{x \to a} f(x) = f(a) \quad \iff \quad \lim_{x \to a} (f(x) - f(a)) = 0$$

$$\iff \quad \lim_{h \to 0} (f(a+h) - f(a)) = 0$$

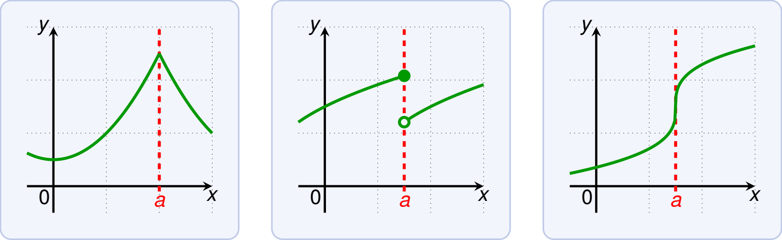
If the latter limit would not be 0 (or not exist),
then $\frac{f(a+h)-f(a)}{h}$ would get arbitrarily large for small h.

If f is continuous at a, then f is not always differentiable at a.

E.g. $|x|$ is continuous at 0 but not differentiable at 0.

How can a Function fail to be Derivable?

There are the following reasons for failure of being derivable:



- graph changes direction abruptly (graph has a “corner”)
- the function is not continuous at a
- graph has a vertical tangent at a, that is:

$$\lim_{x \to a} |f'(x)| = \infty$$

Example for a vertical tangent is $f(x) = \sqrt[3]{x}$ at 0.

Derivative: Other Notations

We usually write $f'(x)$ for the derivative.

However, there are other common notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols $\frac{d}{dx}$ and D are called **differentiation operators.**
(they indicate the operation of computing the derivative)

The notation $\frac{dy}{dx}$ has been introduced by Leibnitz:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

In Leibnitz notation $f'(a)$ is written as

$$\left.\frac{dy}{dx}\right|_a \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_a$$

Higher Derivatives

If f is a function, the derivative f' is also a function.

Thus we can compute the derivative of the derivative:

$$(f')' = f''$$

The function f'' is called **second derivative** of f.

Let $f(x) = x^3 - x$. Find $f''(x)$.

We have seen $f'(x) = 3x^2 - 1$. Thus

$$\begin{aligned}
f''(x) = (f')'(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x
\end{aligned}$$

Higher Derivatives

What is the meaning of $f''(x)$?

- the slope of $f'(x)$ at point $(x, f'(x))$
- the rate of change of $f'(x)$
- the rate of change of the rate of change of $f(x)$

The **acceleration** is an example of a second derivative:

- $s(t)$ is the position of an object (at time t)
- $v(t) = s'(t)$ is the speed (at time t)
- $a(t) = v'(t) = s''(t)$ is the acceleration (at time t)

Higher Derivatives

We can continue this process of deriving:

- $f'''(x) = (f'')'(x)$
- $f''''(x) = (f''')'(x)$
- $\dots$

The n-th derivative of f is denoted by

$$f^{(n)}(x) \qquad \text{or} \qquad \frac{d^n y}{dx^n}$$

For example, $f = f^{(0)}$, $f' = f^{(1)}$, $f'' = f^{(2)}$, $f''' = f^{(3)}$

Let $f(x) = x^3 - x$. Find $f'''(x)$ and $f^{(4)}(x)$.

We know $f''(x) = 6x$. Hence

$$f'''(x) = 6 \qquad f^{(4)}(x) = 0$$

Note that f''' is the slope of f'', and $f^{(4)}$ is the slope of f'''.