Calculus M211

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Indiana University Bloomington

2013

Derivative as a Function

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&= 3x^2 - 1
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Using the definition of derivative, find $f'(x)$, where $f(x) = \sqrt{2x}$.

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&= \lim_{h \to 0} \left(\frac{2}{\sqrt{2x+2h} + \sqrt{2x}} \right)
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\end{aligned}
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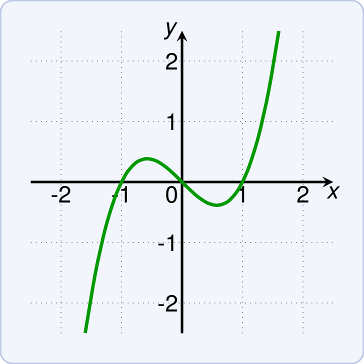
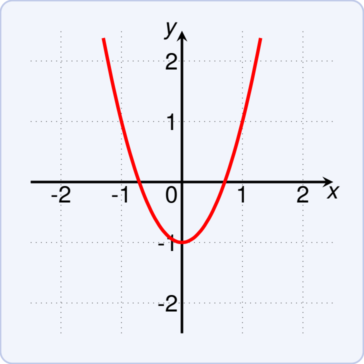
Derivative as a Function

Which of these functions is the derivative of the other?





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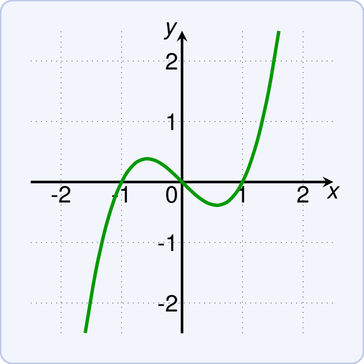

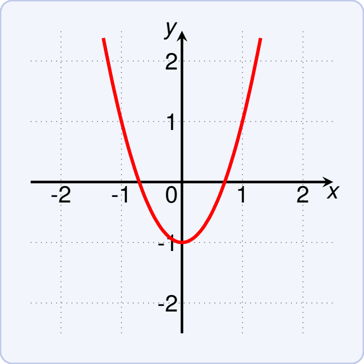

The right is the derivative of the left:

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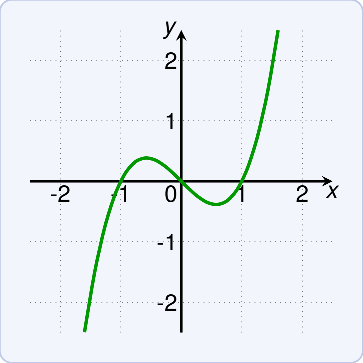

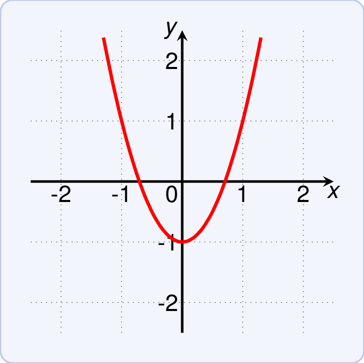

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- look at local maxima and minima of f; then f' must be 0

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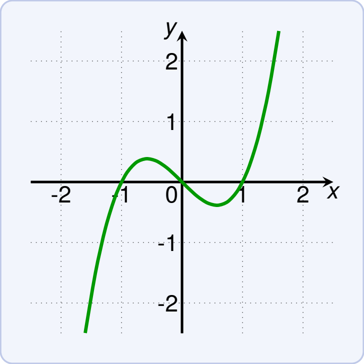

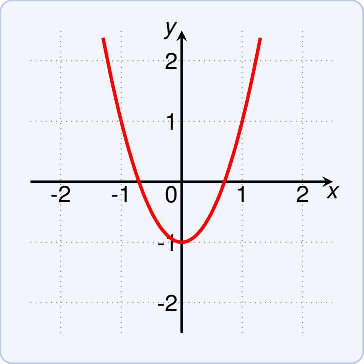

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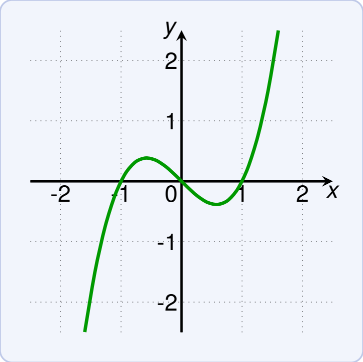

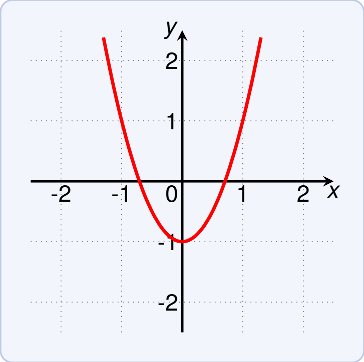

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- where f increases, f' must be positive
- where f decreases, f' must be negative

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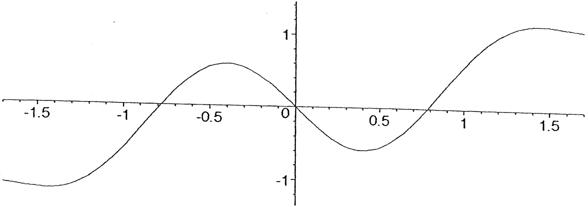

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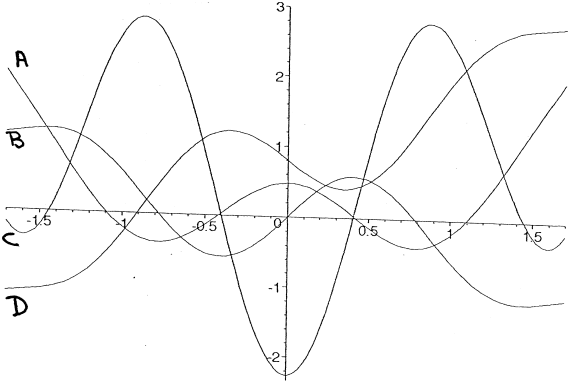

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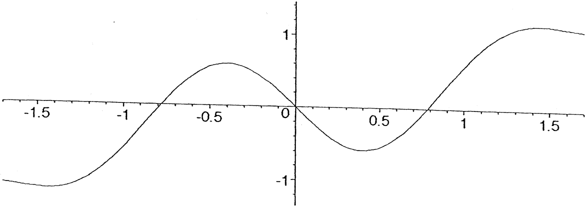

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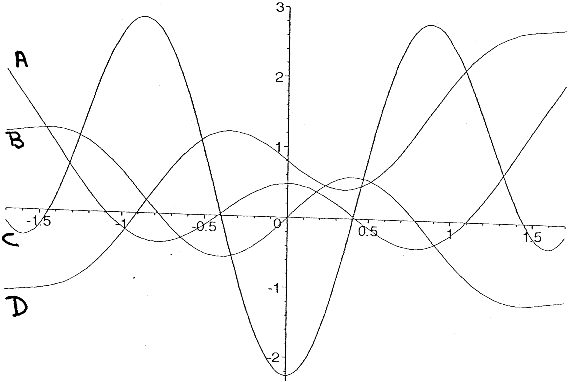

C

Derivative as a Function

Practice computing derivatives!

Try examples from the book for yourself. Among others:

- Example 3 in Section 2.8
- Example 4 in Section 2.8

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Note that the interval (a, b) may be $(-\infty, b)$, (a, ∞) or $(-\infty, \infty)$.

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We need to look at the left and right limits:

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Hence f is differentiable at all numbers in $(-\infty, 0) \cup (0, \infty)$.

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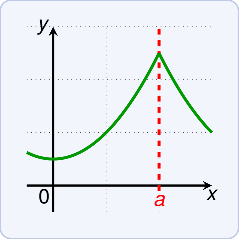
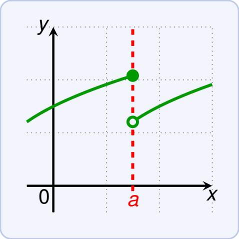
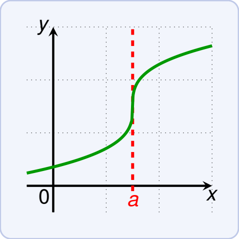
E.g. $|x|$ is continuous at 0 but not differentiable at 0.

How can a Function fail to be Derivable?

There are the following reasons for failure of being derivable:







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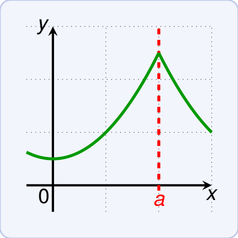

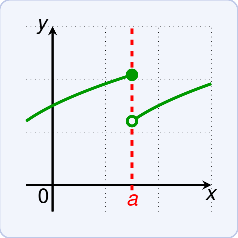

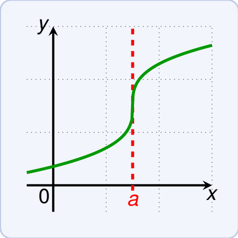

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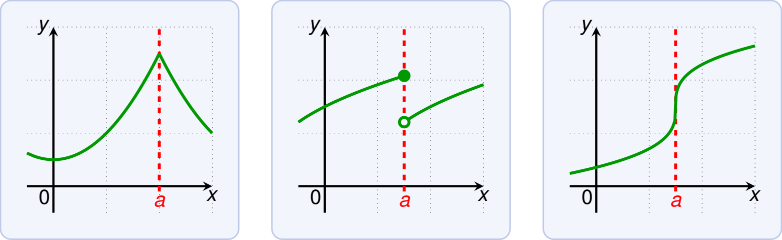

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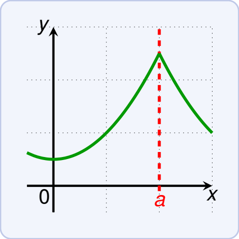

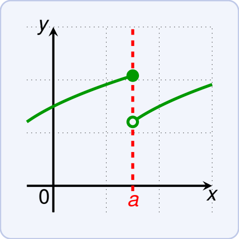

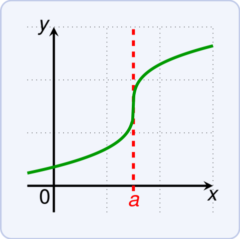

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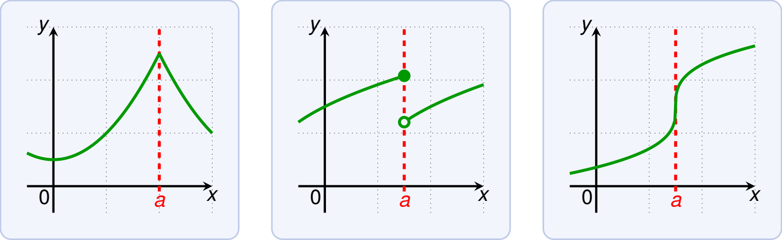

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Example for a vertical tangent is $f(x) = \sqrt[3]{x}$ at 0.

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In Leibnitz notation $f'(a)$ is written as

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For example, $f = f^{(0)}, \quad f' = f^{(1)}, \quad f'' = f^{(2)}, \quad f''' = f^{(3)}$

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For example, $\quad f = f^{(0)}, \quad f' = f^{(1)}, \quad f'' = f^{(2)}, \quad f''' = f^{(3)}$

Let $f(x) = x^3 - x$. Find $f'''(x)$ and $f^{(4)}(x)$.

Higher Derivatives

We can continue this process of deriving:

- $f'''(x) = (f'')'(x)$
- $f''''(x) = (f''')'(x)$
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Note that f''' is the slope of f'', and $f^{(4)}$ is the slope of f'''.