

# Calculus M211

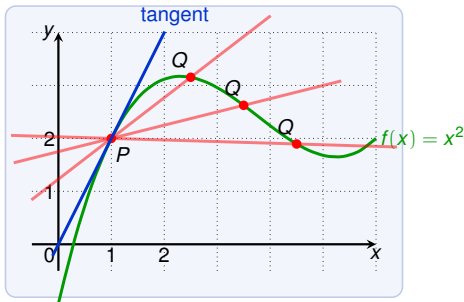
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# Finding a Tangent

We move  $Q$  closer and closer to  $P$ .



The limit is the tangent.

The **tangent line** to the curve  $f(x)$  at point  $P = (a, f(a))$  is the line through  $P$  with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

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provided that the limit exists.

Find an equation of the tangent line to  $f(x) = x^2$  at point  $(1, 1)$ .

We use the equation for the slope with  $a = 1$ :

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2 \end{aligned}$$

Thus  $y - 1 = 2(x - 1)$ , that is, the tangent is  $y = 2x - 1$ .

# Finding a Tangent

Alternative definition of the slope:

The slope of  $f$  at point  $(a, f(a))$  is:

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

The slope  $m$  is also called the **slope of the curve** at the point.

Find an equation of the tangent to  $f(x) = \frac{3}{x}$  at point  $(3, 1)$ .

The slope is:

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{3-(3+h)}{3+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h(3+h)} = \lim_{h \rightarrow 0} -\frac{1}{3+h} = -\frac{1}{3} \end{aligned}$$

Thus  $y - 1 = -\frac{1}{3}(x - 3)$ , that is, the tangent is  $y = 2 - \frac{x}{3}$ .

# Velocities

Let  $f(t)$  be a **position function** of an object:

- ▶  $f(t)$  is the position (distance from the origin) after time  $t$

The average velocity in the time interval  $(a, a + h)$  is

$$\text{average velocity} = \frac{\text{difference in position}}{\text{time difference}} = \frac{f(a + h) - f(a)}{h}$$

The (instantaneous) **velocity**  $v(a)$  at time  $t = a$  is:

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

which is the slope of the tangent at point  $(a, f(a))$ .

Let  $f(t) = 2t^2$ . What is the speed of the object after  $n$  seconds?

$$\begin{aligned} v(n) &= \lim_{h \rightarrow 0} \frac{2 \cdot (n + h)^2 - 2 \cdot n^2}{h} = \lim_{h \rightarrow 0} \frac{4nh + 2 \cdot h^2}{h} \\ &= \lim_{h \rightarrow 0} (4n + 2 \cdot h) = 4n \end{aligned}$$

# Derivatives

The **derivative of a function  $f$  at a number  $a$** , denoted  $f'(a)$ , is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

Find the derivative of  $f(x) = x^2 - 8x + 9$  at number  $a$ .

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \rightarrow 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

# Derivatives

The **derivative of a function  $f$  at a number  $a$** , denoted  $f'(a)$ , is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

An equivalent way of defining the derivative (take  $x = a + h$ ):

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

The tangent line to  $f$  at point  $(a, f(a))$  is the line through  $(a, f(a))$  with slope  $f'(a)$ , the derivative of  $f$  at  $a$ .

Find an equation of the tangent to  $f(x) = x^2 - 8x + 9$  at  $(3, -6)$ .

We know  $f'(a) = 2a - 8$ , and thus  $f'(3) = -2$ .

Hence  $y + 6 = -2(x - 3)$ , that is,  $y = -2x$

# Rates of Change

Suppose  $y$  is a quantity that depends on  $x$ . That is  $y = f(x)$ .

If  $x$  changes from  $x_1$  to  $x_2$ , the change (increment) of  $x$  is

$$\Delta x = x_2 - x_1$$

and the corresponding change in  $y$  is

$$\Delta y = f(x_2) - f(x_1)$$

The **average rate of change over the interval**  $[x_1, x_2]$  is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The **instantaneous rate of change** by letting  $\Delta x$  go to 0:

$$\text{instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This is the derivative  $f'(x_1)$ !

(Note that large derivative  $f'(x_1)$  means rapid change.)



# Rates of Change

A manufacturer produces some fabric. The costs for producing  $x$  yards are  $f(x)$  dollars.

- ▶ What is the meaning of  $f'(x)$  (called **marginal costs**)?
- ▶ What does it mean to say  $f'(1000) = 9$ ?
- ▶ Which do you think is greater  $f'(50)$  or  $f'(500)$ ?

Answers:

- ▶  $f'(x)$  is the rate of change of production costs in dollars per yard with respect to the number of yards produced
- ▶  $f'(1000) = 9$  means that after having produced 1000 yards, the costs increase by 9 dollars for additional yards
- ▶ Typically  $f'(500) < f'(50)$  since usually the cost of production per yard will decrease the more you produce (due to fixed costs: you have already bought and installed the machines. . .)

# Rates of Change

Rates of change are important in:

- ▶ all natural sciences,
- ▶ in engineering, and
- ▶ social sciences

Examples of rate of change:

- ▶ in economics: change of production costs with respect to the number of items produced (called marginal costs)
- ▶ in physics: rate of change of work with respect to time (called power)
- ▶ in chemistry: rate of change of the concentration of a reactant with respect to time (called rate of reaction)
- ▶ in biology: rate of change of the population of bacteria with respect to time