Calculus M211

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1st Midterm Exam - Review

Express the domain of the function

$$f(x) = \frac{x + \log(x+1) + \sqrt{5-x}}{x-2}$$

as a union of intervals.

We analyze the parts:

- $\log(x+1)$ is defined for $x > -1$, thus $(-1, \infty)$
- $\sqrt{5-x}$ is defined on $x \le 5$, thus $(-\infty, 5]$
- the fraction $\frac{\cdots}{x-2}$ is defined for $x \neq 2$, thus $(-\infty, 2) \cup (2, \infty)$

The domain of f is not:

$$(-1, \infty) \cup (-\infty, 5] \cup (-\infty, 2) \cup (2, \infty) \quad = \quad (-\infty, \infty)$$

The domain of f is:

$$(-1, 2) \cup (2, 5]$$

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Find the precise value of

$$\log_5 198 - \log_5 10 - \log_5 99$$

We have

$$\begin{aligned}\log_5 198 - \log_5 10 - \log_5 99 &= \log_5 \frac{198}{10} - \log_5 99 \\ &= \log_5 \frac{198}{10 \cdot 99} \\ &= \log_5 \frac{2}{10} \\ &= \log_5 \frac{1}{5} \\ &= -1\end{aligned}$$

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Find the inverse function of

$$f(x) = \frac{1 + \log x}{2\log x + 5}$$

We have

$$\begin{aligned} y = \frac{1 + \log x}{2\log x + 5} &\implies y \cdot (2\log x + 5) = 1 + \log x \\ &\implies 2y\log x + 5y = 1 + \log x \\ &\implies 2y\log x - \log x = 1 - 5y \\ &\implies \log x \cdot (2y - 1) = 1 - 5y \\ &\implies \log x = \frac{1 - 5y}{2y - 1} \\ &\implies x = 10^{\frac{1-5y}{2y-1}} \end{aligned}$$

Thus the inverse function is $f(y) = 10^{\frac{1-5y}{2y-1}}$.

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Prove or disprove that the following limit exists

$$\lim_{x \to 5} \frac{x-5}{|x-5|}$$

For $x < 5$ we have $\frac{x-5}{|x-5|} = -1$. Thus

$$\lim_{x \to 5^-} \frac{x-5}{|x-5|} = \lim_{x \to 5^-} -1 = -1$$

For $x > 5$ we have $\frac{x-5}{|x-5|} = 1$. Thus

$$\lim_{x \to 5^+} \frac{x-5}{|x-5|} = \lim_{x \to 5^+} 1 = 1$$

The limit $\lim_{x \to 5} \frac{x-5}{|x-5|}$ does not exist since the left- and the right-limit are different.

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Prove that the equation

$$2\cos(\frac{x\pi}{2}) = 2^x$$

has a solution for x on the interval $[0, 1]$.

Define $f(x) = 2\cos(\frac{x\pi}{2}) - 2^x$:

$$2\cos(\frac{x\pi}{2}) = 2^x \qquad \Longleftrightarrow \qquad f(x) = 0$$

We have:

- $f(x)$ is defined on $[0, 1]$
- $f(x)$ is continuous on its domain since it is a composition, product, division and multiplication of continuous functions
- $f(0) = 1$ and $f(1) = -2$

Since 0 is between -2 and 1, by the Intermediate Value Theorem, there exists c in $(0, 1)$ such that $f(c) = 0$. This c is a solution of the equation.

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Prove $\lim_{x \to 0} g(x) = 0$ where $g(x) = x^{12} \cdot \cos\left(\frac{1+e^{50x}}{13.2x^2}\right)$.

We know that the range of cos is $[-1, 1]$. Thus

$$-x^{12} \leq g(x) \leq x^{12}$$

Moreover $\lim_{x \to 0} -x^{12} = 0 = \lim_{x \to 0} x^{12}$.

Thus we can apply the Squeeze Theorem with

- lower bound $-x^{12}$ (i.e. $\leq g(x)$), and
- upper bound x^{12} (i.e. $\geq g(x)$)

and it follows that

$$\lim_{x \to 0} g(x) = 0$$

1st Midterm Exam - Review

For what value of k is the following function continuous?

$$f(x) = \begin{cases} x^2 + 2k & \text{for } x < 2 \\ 3^x - k & \text{for } x \geq 2 \end{cases}$$

For any k, the function is continuous at all $x \neq 2$ since

- $x^2 + 2k$ is continuous, and
- $3^x - k$ is continuous.

(Both are compositions of continuous functions)

At point $x = 2$ we have:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + 2k = 4 + 2k$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3^x - k = 9 - k$$

$$f(2) = 3^2 - k = 9 - k$$

We have continuity at 2 if $4 + 2k = 9 - k$. Thus $k = \frac{5}{3}$.

Departmental Help Sessions

- 6:00 - 8:00 PM in WY 115 on Monday, Tuesday, Wednesday and Thursday.