Calculus M211

Jörg Endrullis

Indiana University Bloomington

2013

Continuity

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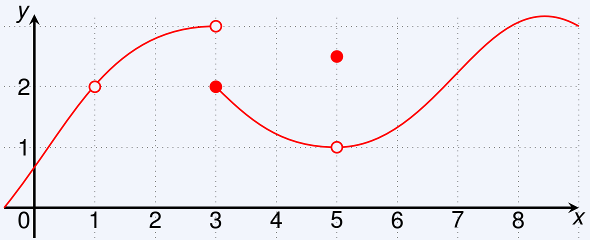
A function f is **discontinuous** at a number a if

- f is defined near a (except perhaps a), and
- f is not continuous at a

Continuity: Examples



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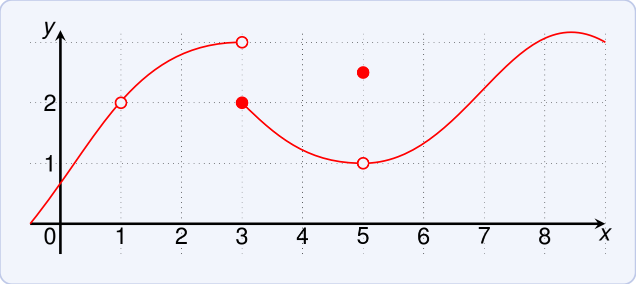

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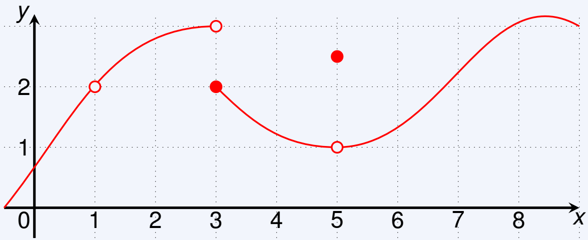

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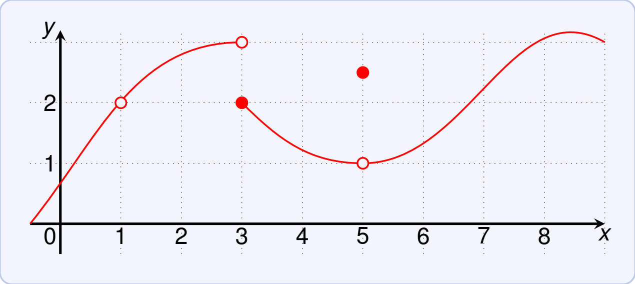

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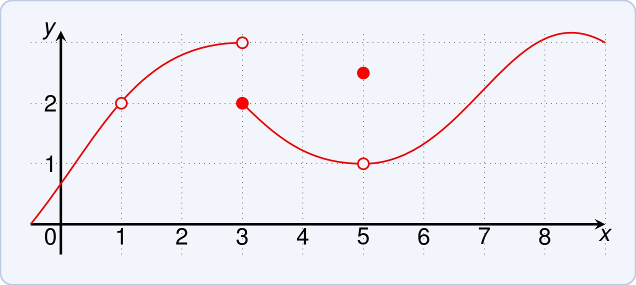

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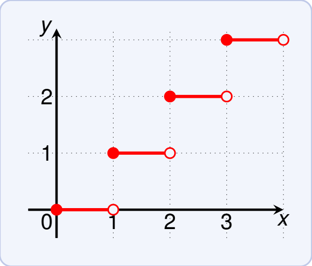
$\lfloor x \rfloor =$ ‘ the largest integer $\leq x$ ’



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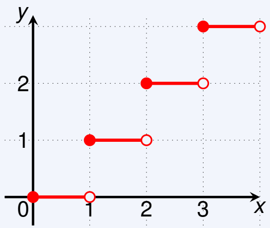

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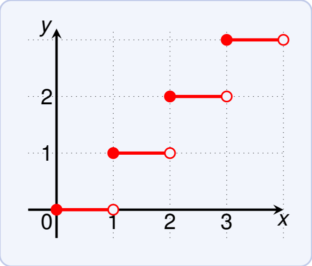

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- but right-continuous everywhere
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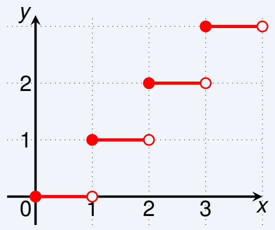

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Therefore f is continuous on $[-1, 1]$.

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Thus $f + g$ is continuous at a.

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(inverse) trigonometric   exponential   logarithmic

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Then their sum and quotient are continuous on their domain.
The domain contains π, so: $\lim_{x \to \pi} f(x) = f(\pi) = 0/(2 - 1) = 0$.

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$$\lim_{x \to 4} \sin(\frac{\pi}{4+\sqrt{x}}) = \sin(\lim_{x \to 4} \frac{\pi}{4+\sqrt{x}}) \quad \text{since sin is continuous}$$

$$= \sin(\frac{\pi}{4+\sqrt{4}}) \quad \text{direct substitution}$$

$$= \sin(\frac{\pi}{6})$$

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$$\begin{aligned}
\lim_{x\to 4} \sin(\frac{\pi}{4+\sqrt{x}}) &= \sin(\lim_{x\to 4} \frac{\pi}{4+\sqrt{x}}) && \text{since sin is continuous} \\
&= \sin(\frac{\pi}{4+\sqrt{4}}) && \text{direct substitution} \\
&= \sin(\frac{\pi}{6}) = \frac{1}{2}
\end{aligned}$$

Continuity: Function Composition

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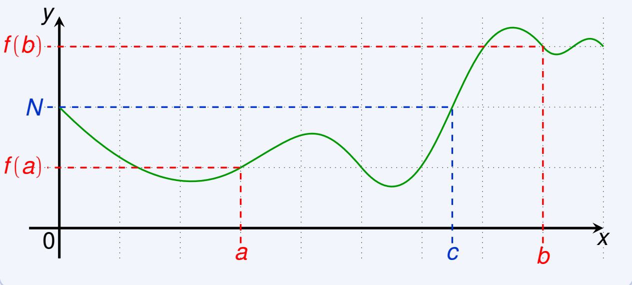
Thus $h(x)$ is continuous on its domain: $\mathbb{R} \setminus \{\pm\pi, \pm 3\pi, \pm 5\pi, \ldots\}$.

Continuity: Intermediate Value Theorem

Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. If N is strictly between $f(a)$ and $f(b)$. Then

$$f(c) = N \qquad \text{for some number } c \text{ in } (a, b)$$



Every N between $f(a)$ and $f(b)$ occurs at least once on (a, b).

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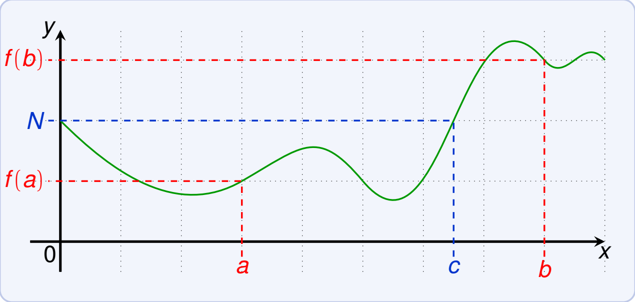

Every N between $f(a)$ and $f(b)$ occurs at least once on (a, b).
Intuitively: the graph cannot jump over the line $y = N$.

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Show that there is a root of the equation

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- $f(2) = 4 \cdot 8 - 6 \cdot 4 + 3 \cdot 2 - 2 = 12$

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Hence there exists c in $(1, 2)$ such that $f(c) = 0$.

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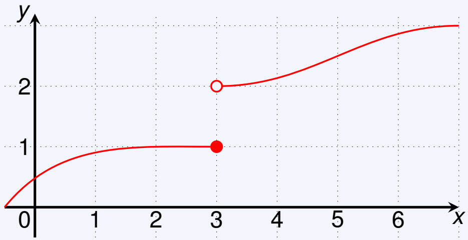

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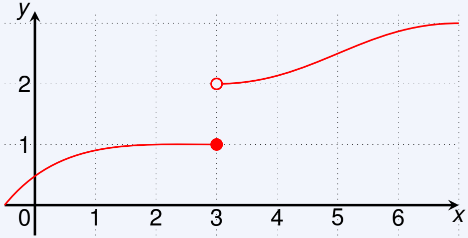

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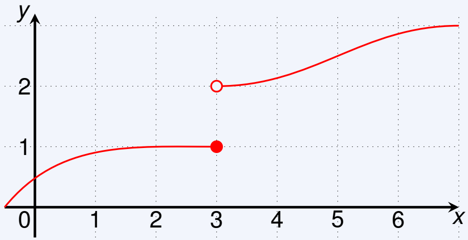

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But there exists no $2 < c < 4$ such that $f(c) = 1.5$!

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By the Intermediate Value Theorem there exists x in the interval $[0, 1]$ such that $f(x) = 0$. This x is a solution of the equation.