Calculus M211

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Precise Definition of Limits

Recall the definition of limits:

Suppose $f(x)$ is defined close to a (but not necessarily a itself). We write

$$\lim_{x \to a} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches a, is L"

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

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The intuitive definition of limits is for some purposes too vague:

- What means 'make $f(x)$ arbitrarily close to L' ?
- What means 'taking x sufficiently close to a' ?

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

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Thus $|f(x) - 5| < 0.1$ whenever $0 < |x - 3| < 0.05$

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We have derived

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$$|f(x) - 5| < 0.1 \quad \text{whenever} \quad 0 < |x - 3| < 0.05$$

In words this means:

If x is within a distance of 0.05 from 3 (and $x \neq 3$) then $f(x)$ is within a distance of 0.1 from 5.

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Similarly, we find

$$|f(x) - 5| < 0.1 \quad \text{whenever} \quad 0 < |x - 3| < 0.05$$
$$|f(x) - 5| < 0.01 \quad \text{whenever} \quad 0 < |x - 3| < 0.005$$
$$|f(x) - 5| < 0.001 \quad \text{whenever} \quad 0 < |x - 3| < 0.0005$$

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The distances 0.1, 0.01, ... are called **error tolerance.**

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Similarly, we find

$$\begin{aligned} |f(x) - 5| < 0.1 \quad &\text{whenever} \quad 0 < |x - 3| < \delta(0.1) \\ |f(x) - 5| < 0.01 \quad &\text{whenever} \quad 0 < |x - 3| < 0.005 \\ |f(x) - 5| < 0.001 \quad &\text{whenever} \quad 0 < |x - 3| < 0.0005 \end{aligned}$$

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Thus **$\delta(\epsilon)$ is a function of the error tolerance ϵ**!

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We need to define $\delta(\epsilon)$ for arbitrary error tolerance $\epsilon > 0$:

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We want $|f(x) - 5| = 2|x - 3| < \epsilon$.

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The distances 0.1, 0.01, $\dots$ are called **error tolerance.**

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We need to define $\delta(\epsilon)$ for arbitrary error tolerance $\epsilon > 0$:

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We want $|f(x) - 5| = 2|x - 3| < \epsilon$. We define $\delta(\epsilon) = \epsilon/2$.

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We define $\delta(\epsilon) = \epsilon/2$. Then the following holds

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In words this means:

> If x is within a distance of $\epsilon/2$ from 3 (and $x \neq 3$)
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We can make ϵ arbitrarily small (but greater 0),
and thereby make $f(x)$ arbitrarily close 5.

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We can make ϵ arbitrarily small (but greater 0),
and thereby make $f(x)$ arbitrarily close 5.

This motivates the precise definition of limits…

Precise Definition of Limits

Let f be a function that is defined on some open interval that contains a, except possibly on a itself.

$$\lim_{x \to a} f(x) = L$$

if there exists a function $\delta : (0, \infty) \to (0, \infty)$ s.t. for every $\epsilon > 0$:

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$$\text{if} \quad 0 < |a - x| < \delta(\epsilon) \quad \text{then} \quad |f(x) - L| < \epsilon$$

In words: No matter what $\epsilon > 0$ we choose,
if the distance of x to a is smaller than $\delta(\epsilon)$ (and $x \neq a$)
then the distance of $f(x)$ to L is smaller than ϵ.

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We can make f arbitrarily close to L by taking ϵ arbitrarily small.

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In words: No matter what $\epsilon > 0$ we choose,

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then the distance of $f(x)$ to L is smaller than ϵ.

We can make f arbitrarily close to L by taking ϵ arbitrarily small.

Then x is sufficiently close to a if the distance is $< \delta(\epsilon)$.

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Let f be a function that is defined on some open interval that contains a, except possibly on a itself.

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if there exists a function $\delta : (0, \infty) \to (0, \infty)$ s.t. for every $\epsilon > 0$:

$$\text{if} \quad 0 < |a - x| < \delta(\epsilon) \quad \text{then} \quad |f(x) - L| < \epsilon$$

The definition is **equivalent to the one in the book**:

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

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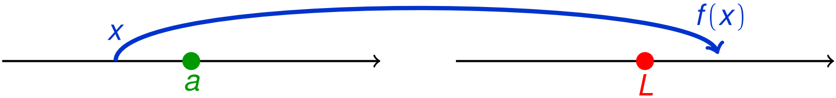
Geometric interpretation:



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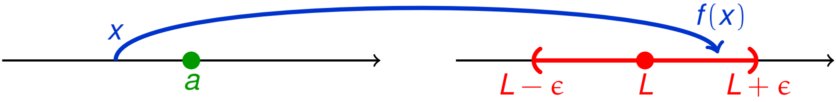
Geometric interpretation:

For any small interval $(L - \epsilon, L + \epsilon)$ around L,



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Geometric interpretation:

For any small interval $(L - \epsilon, L + \epsilon)$ around L,
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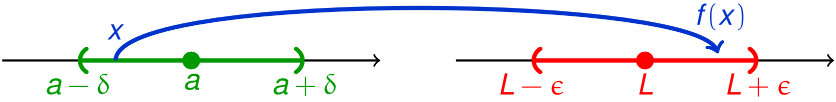

Precise Definition of Limits

$$\lim_{x \to a} f(x) = L$$

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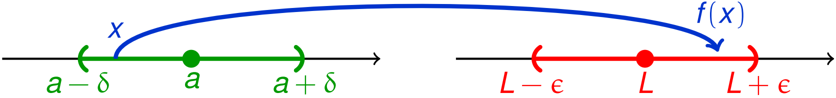

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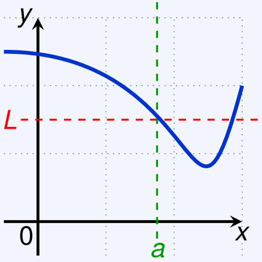

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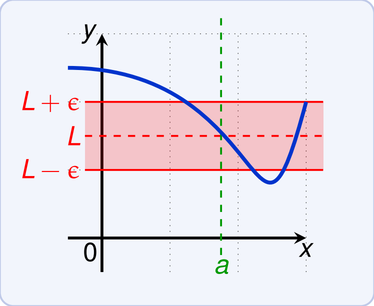

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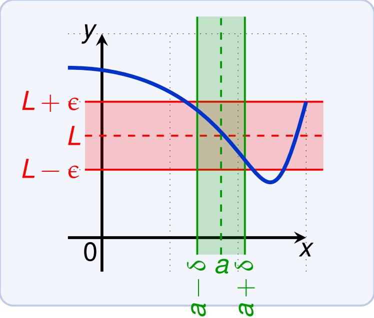

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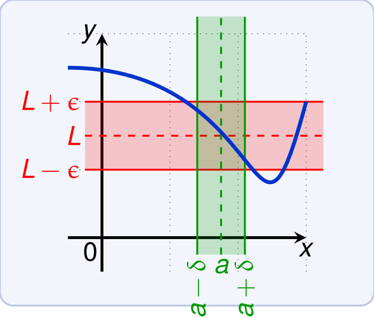

For every interval I_L around L,

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such that

if we restrict the domain of f to I_a, then the curve lies in I_L.

Precise Definition of Limits - Example

Proof that

$$\lim_{x \to 3} (4x - 5) = 7$$

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Thus $\delta = \frac{\epsilon}{4}$.

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Let $\epsilon > 0$ be arbitrary (the error tolerance).

We need to find δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \epsilon$$

We have

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Thus $\delta = \frac{\epsilon}{4}$. If $0 < |x - 3| < \frac{\epsilon}{4}$ then $|(4x - 5) - 7| < \epsilon$.

Precise Definition of Limits - Example

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In exams many students write:

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which is wrong.

Correct is:

If $0 < |x - 3| < \frac{\epsilon}{4}$

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Precise Definition of Limits - Example

Find $\delta > 0$ such that

$$\text{if} \quad 0 < |x-1| < \delta \quad \text{then} \quad |(x^2 - 5x + 6) - 2| < 0.2$$

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$$(x^2 - 5x + 6) - 2 = (1 + \delta)^2 - 5(1 + \delta) + 4$$

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Thus

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Thus: $\quad$ if $\quad 4|\delta| < 0.2 \quad$ then $\quad |(x^2 - 5x + 6) - 2| < 0.2$.

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Thus: $\quad$ if $\quad 4|\delta| < 0.2 \quad$ then $\quad |(x^2 - 5x + 6) - 2| < 0.2$.

Hence $\delta = 0.04$ is a possible choice.

Precise Definition of Limits: Example

Let $\lim_{x\to a} f(x) = L_f$ and $\lim_{x\to a} g(x) = L_g$. Prove the sum law:

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Let $\epsilon > 0$ be arbitrary, we need to find δ such that

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We know that there exists δ_f such that:

$$\text{if} \quad 0 < |x - a| < \delta_f \quad \text{then} \quad |f(x) - L_f| < \epsilon/2$$

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We take $\delta = \min(\delta_f, \delta_g)$.

Precise Definition of Limits: Example

Let $\lim_{x\to a} f(x) = L_f$ and $\lim_{x\to a} g(x) = L_g$. Prove the sum law:

$$\lim_{x\to a} [f(x) + g(x)] = L_f + L_g$$

Let $\epsilon > 0$ be arbitrary, we need to find δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |(f(x) + g(x)) - (L_f + L_g)| < \epsilon$$

Note that $(f(x) + g(x)) - (L_f + L_g) \;=\; (f(x) - L_f) + (g(x) - L_g)$.

We know that there exists δ_f such that:

$$\text{if} \quad 0 < |x - a| < \delta_f \quad \text{then} \quad |f(x) - L_f| < \epsilon/2$$

and there exists δ_g such that:

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$$|f(x) - L_f| < \epsilon/2 \qquad \text{and} \qquad |g(x) - L_g| < \epsilon/2$$

and hence $|(f(x) - L_f) + (g(x) - L_g)| < \epsilon$.

Precise Definition of One-Sided Limits

Left-limit

$$\lim_{x \to a^-} f(x) = L$$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \epsilon$$

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Right-limit

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that

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Precise Definition of One-Sided Limits - Example

Right-limit

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Thus $\delta = \epsilon^2$.

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We have (since $0 < x$)

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Thus $\delta = \epsilon^2$. If $0 < x < 0 + \epsilon^2$ then $|\sqrt{x} - 0| < \epsilon$.

Precise Definition of Infinite Limits

Infinite Limit

$$\lim_{x \to a} f(x) = \infty$$

if for every positive number M there is $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad f(x) > M$$

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Negative Infinite Limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every negative number M there is $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad f(x) < M$$

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$$\frac{1}{x^2} > M \quad \iff \quad 1 > M \cdot x^2$$

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Thus $\delta = \sqrt{1/M}$.

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Thus $\delta = \sqrt{1/M}$. If $\quad 0 < |0 - x| < \sqrt{1/M} \quad$ then $\quad \frac{1}{x^2} > M$.