

# Calculus M211

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# Calculating Limits using Limit Laws

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- $$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

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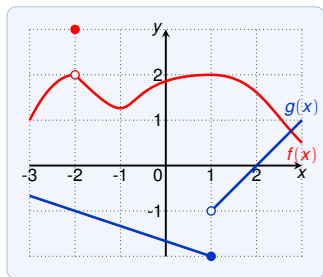
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These laws also work for one-sided limits  $\lim_{x \rightarrow a^\pm}$ .

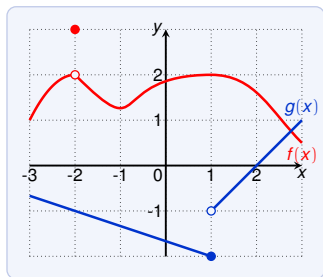
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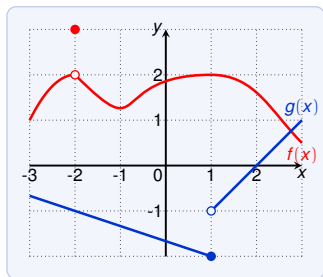
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Use these graphs to estimate:

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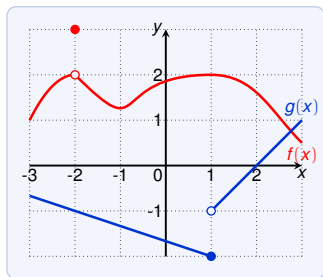


Use these graphs to estimate:

1.  $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$

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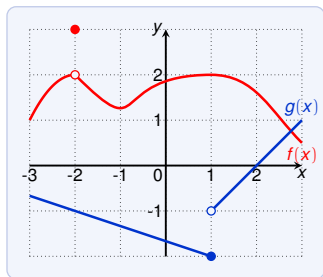


Use these graphs to estimate:

1.  $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$   
 $= \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [5g(x)]$

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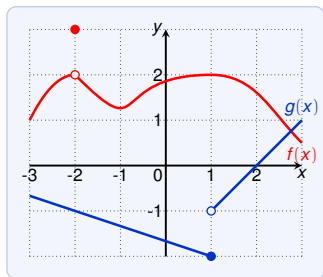


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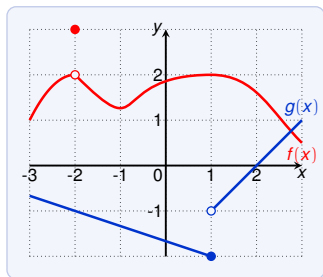
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 $= 2 + 5(-1)$



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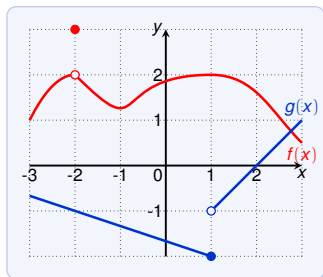


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 $= 2 + 5(-1)$   
 $= -3$

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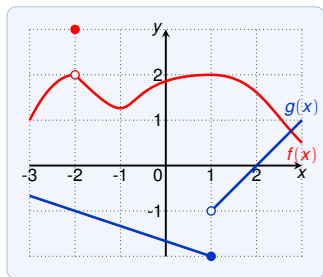


Use these graphs to estimate:

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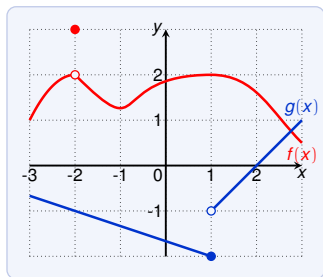


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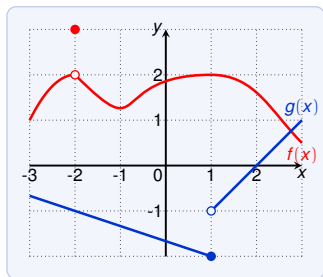


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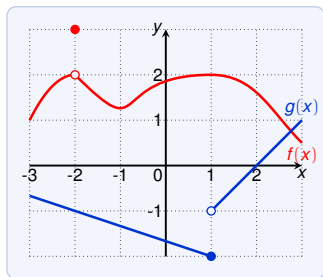


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 $= \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x)$   
 $\leftarrow \lim_{x \rightarrow 1} g(x)$  does not exist  
(we cannot use the limit laws)

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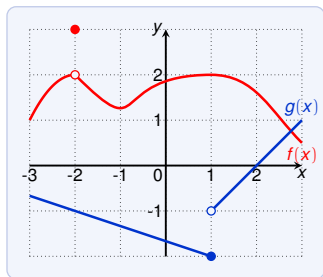


Use these graphs to estimate:

2a.  $\lim_{x \rightarrow 1^-} [f(x)g(x)]$

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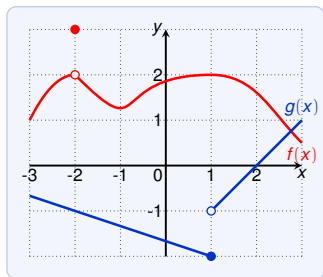


Use these graphs to estimate:

$$\begin{aligned} 2a. \quad \lim_{x \rightarrow 1^-} [f(x)g(x)] \\ = \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) \end{aligned}$$

# Calculating Limits using Limit Laws

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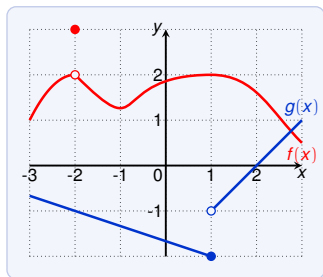
Use these graphs to estimate:

$$\begin{aligned} 2a. \quad & \lim_{x \rightarrow 1^-} [f(x)g(x)] \\ &= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) \\ &= 2 \cdot -2 \end{aligned}$$



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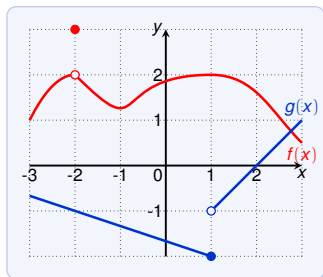


Use these graphs to estimate:

$$\begin{aligned} 2a. \quad & \lim_{x \rightarrow 1^-} [f(x)g(x)] \\ &= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) \\ &= 2 \cdot -2 = -4 \end{aligned}$$

# Calculating Limits using Limit Laws

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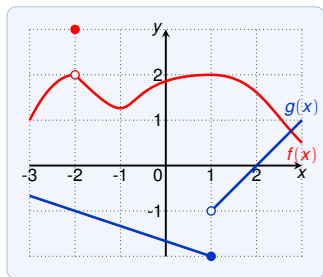


Use these graphs to estimate:

- 2a.  $\lim_{x \rightarrow 1^-} [f(x)g(x)]$   
 $= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x)$   
 $= 2 \cdot -2 = -4$
- 2b.  $\lim_{x \rightarrow 1^+} [f(x)g(x)]$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
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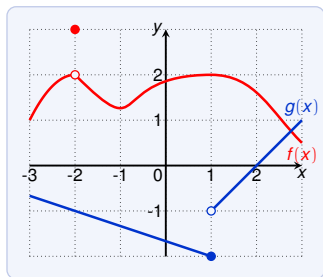


Use these graphs to estimate:

- 2a.  $\lim_{x \rightarrow 1^-} [f(x)g(x)]$   
 $= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x)$   
 $= 2 \cdot -2 = -4$
- 2b.  $\lim_{x \rightarrow 1^+} [f(x)g(x)]$   
 $= \lim_{x \rightarrow 1^+} f(x) \cdot \lim_{x \rightarrow 1^+} g(x)$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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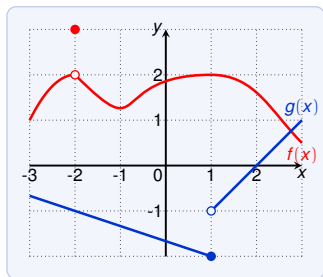
Use these graphs to estimate:

2a.  $\lim_{x \rightarrow 1^-} [f(x)g(x)]$   
 $= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x)$   
 $= 2 \cdot -2 = -4$

2b.  $\lim_{x \rightarrow 1^+} [f(x)g(x)]$   
 $= \lim_{x \rightarrow 1^+} f(x) \cdot \lim_{x \rightarrow 1^+} g(x)$   
 $= 2 \cdot -1$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
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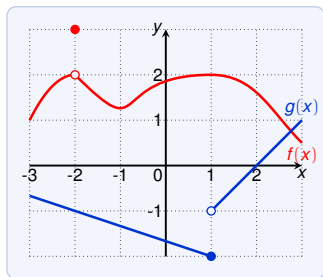
Use these graphs to estimate:

$$\begin{aligned} 2a. \quad \lim_{x \rightarrow 1^-} [f(x)g(x)] &= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) \\ &= 2 \cdot -2 = -4 \end{aligned}$$

$$\begin{aligned} 2b. \quad \lim_{x \rightarrow 1^+} [f(x)g(x)] &= \lim_{x \rightarrow 1^+} f(x) \cdot \lim_{x \rightarrow 1^+} g(x) \\ &= 2 \cdot -1 = -2 \end{aligned}$$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



Use these graphs to estimate:

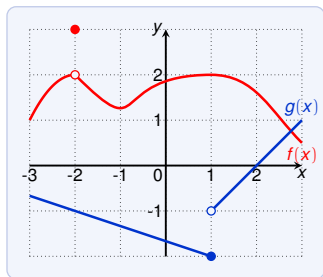
$$\begin{aligned} 2a. \quad \lim_{x \rightarrow 1^-} [f(x)g(x)] &= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) \\ &= 2 \cdot -2 = -4 \end{aligned}$$

$$\begin{aligned} 2b. \quad \lim_{x \rightarrow 1^+} [f(x)g(x)] &= \lim_{x \rightarrow 1^+} f(x) \cdot \lim_{x \rightarrow 1^+} g(x) \\ &= 2 \cdot -1 = -2 \end{aligned}$$

$\Rightarrow \lim_{x \rightarrow 1} [f(x)g(x)]$  does not exist

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$

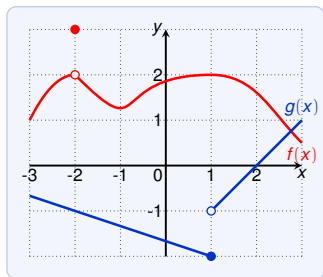


Use these graphs to estimate:

3.  $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



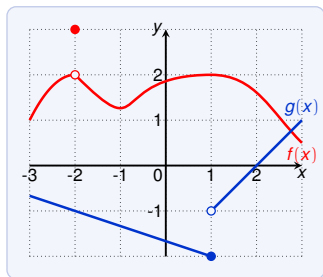
Use these graphs to estimate:

3.  $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 2} f(x)}{\lim_{x \rightarrow 2} g(x)}$



# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
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5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



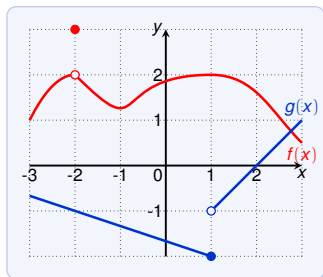
Use these graphs to estimate:

$$3. \lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 2} f(x)}{\lim_{x \rightarrow 2} g(x)}$$

⚡  $\lim_{x \rightarrow 2} g(x) = 0$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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Use these graphs to estimate:

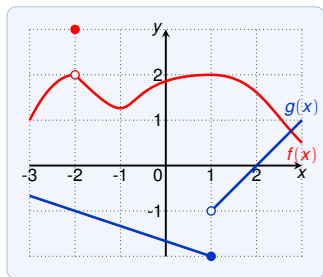
$$3. \lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 2} f(x)}{\lim_{x \rightarrow 2} g(x)}$$

$\leftarrow \lim_{x \rightarrow 2} g(x) = 0$

(we cannot use the limit laws)

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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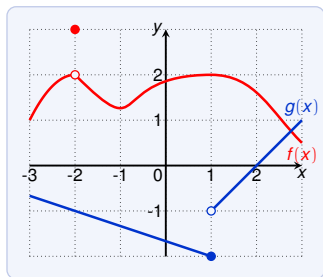
Use these graphs to estimate:

Lets try without limit laws:

3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)}$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



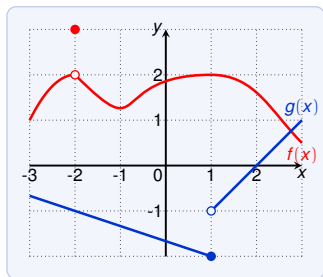
Use these graphs to estimate:

Lets try without limit laws:

3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)} = -\infty$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
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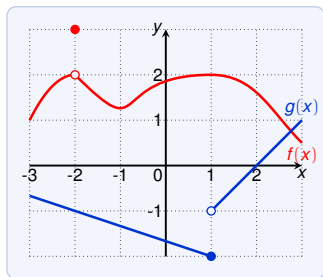
Use these graphs to estimate:

Lets try without limit laws:

- 3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)} = -\infty$   
since  $\lim_{x \rightarrow 2^-} f(x) \approx 1.6$ , and  
 $g(x)$  approaches 0,  $g(x) < 0$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
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3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
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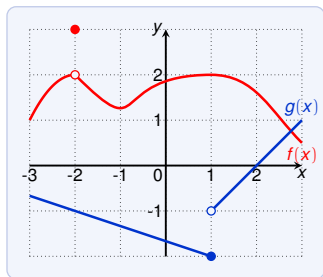
Use these graphs to estimate:

Lets try without limit laws:

- 3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)} = -\infty$   
since  $\lim_{x \rightarrow 2^-} f(x) \approx 1.6$ , and  
 $g(x)$  approaches 0,  $g(x) < 0$
- 3b.  $\lim_{x \rightarrow 2^+} \frac{f(x)}{g(x)}$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



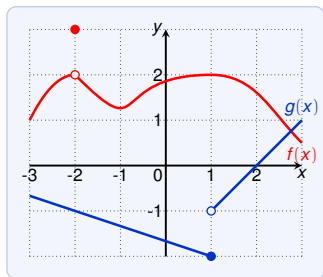
Use these graphs to estimate:

Lets try without limit laws:

- 3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)} = -\infty$   
since  $\lim_{x \rightarrow 2^-} f(x) \approx 1.6$ , and  $g(x)$  approaches 0,  $g(x) < 0$
- 3b.  $\lim_{x \rightarrow 2^+} \frac{f(x)}{g(x)} = \infty$

# Calculating Limits using Limit Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$



Use these graphs to estimate:

Lets try without limit laws:

- 3a.  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)} = -\infty$   
since  $\lim_{x \rightarrow 2^-} f(x) \approx 1.6$ , and  $g(x)$  approaches 0,  $g(x) < 0$
- 3b.  $\lim_{x \rightarrow 2^+} \frac{f(x)}{g(x)} = \infty$   
since  $\lim_{x \rightarrow 2^+} f(x) \approx 1.6$ , and  $g(x)$  approaches 0,  $g(x) > 0$



## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$

## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
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5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$
6.  $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$  for  $n$  a positive integer

## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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6.  $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$  for  $n$  a positive integer
7.  $\lim_{x \rightarrow a} c = c$

## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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6.  $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$  for  $n$  a positive integer
7.  $\lim_{x \rightarrow a} c = c$
8.  $\lim_{x \rightarrow a} x^n = a^n$

## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2.  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
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## More Limits Laws

1.  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
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# Limit Laws: Examples

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$$\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$$

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$$\begin{aligned} \lim_{x \rightarrow 5} (2x^2 - 3x + 4) &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{(law 1 and 2)} \\ &= 2 \lim_{x \rightarrow 5} (x^2) - 3 \lim_{x \rightarrow 5} (x) + 4 && \text{(law 3 and 7)} \end{aligned}$$

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## Limit Laws: Examples

$$\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

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$$\begin{aligned} & \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \\ &= \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} \end{aligned}$$

(law 5)

# Limit Laws: Examples

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# Computing Limits: Direct Substitution Property

## Direct Substitution Property

If  $f$  is a polynomial or a rational and  $a$  is in the domain of  $f$ , then:

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If  $f(x) = g(x)$  for all  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$   
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As a consequence

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} x + 1$$

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## Function Replacement

If  $f(x) = g(x)$  for all  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$   
(provided that the limit exists).

Actually it suffices  $f(x) = g(x)$  when  $x$  is close to  $a$ .

Find  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ .

- ▶ Direct substitution is not applicable because  $x = 1$  is not in the domain.

We replace the function:

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} \text{ for } x \neq 1 = x + 1$$

As a consequence

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} x + 1 = 1 + 1 = 2$$

# Computing Limits: Function Replacement

Find  $\lim_{x \rightarrow 1} g(x)$  where

$$g(x) = \begin{cases} 2x + 1 & \text{for } x \neq 1, \\ \pi & \text{for } x = 1 \end{cases}$$

# Computing Limits: Function Replacement

Find  $\lim_{x \rightarrow 1} g(x)$  where

$$g(x) = \begin{cases} 2x + 1 & \text{for } x \neq 1, \\ \pi & \text{for } x = 1 \end{cases}$$

We have:

$$g(x) = 2x + 1 \quad \text{for all } x \neq 1$$



# Computing Limits: Function Replacement

Find  $\lim_{x \rightarrow 1} g(x)$  where

$$g(x) = \begin{cases} 2x + 1 & \text{for } x \neq 1, \\ \pi & \text{for } x = 1 \end{cases}$$

We have:

$$g(x) = 2x + 1 \quad \text{for all } x \neq 1$$

As a consequence:

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} 2x + 1$$

# Computing Limits: Function Replacement

Find  $\lim_{x \rightarrow 1} g(x)$  where

$$g(x) = \begin{cases} 2x + 1 & \text{for } x \neq 1, \\ \pi & \text{for } x = 1 \end{cases}$$

We have:

$$g(x) = 2x + 1 \quad \text{for all } x \neq 1$$

As a consequence:

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} 2x + 1 = 2 + 1 = 3$$

# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$$

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We have:

$$\frac{(3 + h)^2 - 9}{h}$$

# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

We have:

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h}$$

# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$$

We have:

$$\frac{(3 + h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h}$$

# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

We have:

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} \stackrel{\text{for } h \neq 0}{=} 6 + h$$

# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

We have:

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} \stackrel{\text{for } h \neq 0}{=} 6 + h$$

As a consequence:

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h)$$



# Computing Limits: Function Replacement

Find

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

We have:

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} \stackrel{\text{for } h \neq 0}{=} 6 + h$$

As a consequence:

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

# Computing Limits: Function Replacement

Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

# Computing Limits: Function Replacement

Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

We have:

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# Computing Limits: Function Replacement

Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

We have:

$$\frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3}$$

# Computing Limits: Function Replacement

Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

We have:

$$\frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} = \frac{t^2 + 9 - 9}{t^2 \cdot (\sqrt{t^2 + 9} + 3)}$$

# Computing Limits: Function Replacement

Find

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

We have:

$$\begin{aligned} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} = \frac{t^2 + 9 - 9}{t^2 \cdot (\sqrt{t^2 + 9} + 3)} \\ &= \frac{t^2}{t^2 \cdot (\sqrt{t^2 + 9} + 3)} \end{aligned}$$

# Computing Limits: Function Replacement

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# Computing Limits: Function Replacement

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We have:

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As a consequence:

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3}$$



# Computing Limits: Function Replacement

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We have:

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$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} \\ &= \frac{1}{\sqrt{\lim_{t \rightarrow 0} (t^2 + 9)} + 3} \quad \text{by laws 5, 1, 9, 7} \end{aligned}$$

# Computing Limits: Function Replacement

Find

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# Computing Limits: Function Replacement

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As a consequence:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} \\ &= \frac{1}{\sqrt{\lim_{t \rightarrow 0} (t^2 + 9)} + 3} \quad \text{by laws 5, 1, 9, 7} \\ &= \frac{1}{\sqrt{9} + 3} = \frac{1}{6} \end{aligned}$$

# Limits and One-Sided Limits

We recall the following theorem:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$$

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The theorem in words:

- ▶ The limit of  $f(x)$ , for  $x$  approaching  $a$ , is  $L$  if and only if the left-limit and the right-limit at  $a$  are both  $L$ .

# Limits and One-Sided Limits

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# Limits and One-Sided Limits

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The limit laws also apply for one-sided limits!

- ▶ if  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$   
then  $\lim_{x \rightarrow a} f(x)$  does not exist

# Computing Limits: Function Replacement

Function replacement for one-sided limits:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .



# Computing Limits: Function Replacement

Function replacement for one-sided limits:

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Find  $\lim_{x \rightarrow 2^-} g(x)$  where

$$g(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 5x + 1 & \text{for } x \geq 2 \end{cases}$$

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$$g(x) = x^2 \quad \text{for all } x < 2$$

# Computing Limits: Function Replacement

Function replacement for one-sided limits:

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We have

$$g(x) = x^2 \quad \text{for all } x < 2$$

Hence:

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} x^2$$

# Computing Limits: Function Replacement

Function replacement for one-sided limits:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Find  $\lim_{x \rightarrow 2^-} g(x)$  where

$$g(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 5x + 1 & \text{for } x \geq 2 \end{cases}$$

We have

$$g(x) = x^2 \quad \text{for all } x < 2$$

Hence:

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} x^2 = 4$$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since  $|x| = x$  for all  $x > 0$  we obtain:

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x$$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since  $|x| = x$  for all  $x > 0$  we obtain:

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$



# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since  $|x| = x$  for all  $x > 0$  we obtain:

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

Since  $|x| = -x$  for all  $x < 0$  we obtain:

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} -x$$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since  $|x| = x$  for all  $x > 0$  we obtain:

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

Since  $|x| = -x$  for all  $x < 0$  we obtain:

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} -x = 0$$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Find  $\lim_{x \rightarrow 0} |x|$  where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since  $|x| = x$  for all  $x > 0$  we obtain:

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

Since  $|x| = -x$  for all  $x < 0$  we obtain:

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} -x = 0$$

Hence  $\lim_{x \rightarrow 0} |x| = 0$ .

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

For all  $x > 0$  we have  $\frac{|x|}{x}$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

For all  $x > 0$  we have  $\frac{|x|}{x} = \frac{x}{x}$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

For all  $x > 0$  we have  $\frac{|x|}{x} = \frac{x}{x} = 1$ .

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

For all  $x > 0$  we have  $\frac{|x|}{x} = \frac{x}{x} = 1$ . Thus

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$



# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

For all  $x < 0$  we have  $\frac{|x|}{x}$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

For all  $x > 0$  we have  $\frac{|x|}{x} = \frac{x}{x} = 1$ . Thus

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

For all  $x < 0$  we have  $\frac{|x|}{x} = \frac{-x}{x}$

# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

If  $f(x) = g(x)$  for all  $x > a$ , then  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x)$ .

Proof that  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist.

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$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

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# Computing Limits: Function Replacement

For one-sided limits we have:

If  $f(x) = g(x)$  for all  $x < a$ , then  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} g(x)$ .

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Hence  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist since  $\lim_{x \rightarrow 0^-} \frac{|x|}{x} \neq \lim_{x \rightarrow 0^+} \frac{|x|}{x}$ .

# Properties of Limits

If

- ▶  $f(x) \leq g(x)$  when  $x$  is near  $a$  (except possibly  $a$ ),
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As a consequence:

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for all  $a \in (-1, 1)$ .

# Properties of Limits

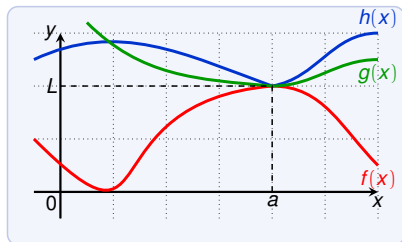
## The Squeeze Theorem

If  $f(x) \leq g(x) \leq h(x)$  when  $x$  is near  $a$  (except possibly  $a$ ) and

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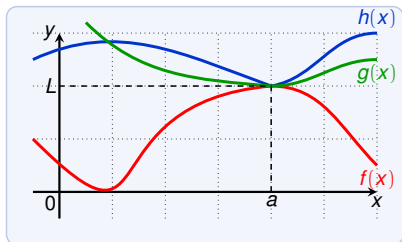
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Here  $f$  is below  $g$ , and  $h$  is above  $g$  (close to  $a$ ). If  $f$  and  $h$  have the same limit, then the squeezed function  $g$  also has.

# Properties of Limits

Show that  $\lim_{x \rightarrow 0} g(x) = 0$  where  $g(x) = x^2 \cdot \sin \frac{1}{x}$ .

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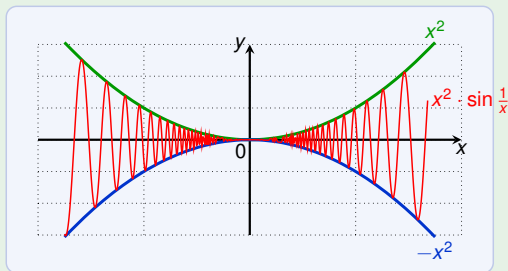
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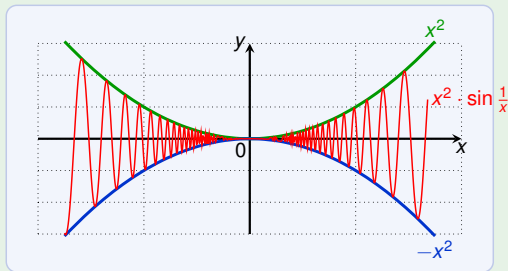


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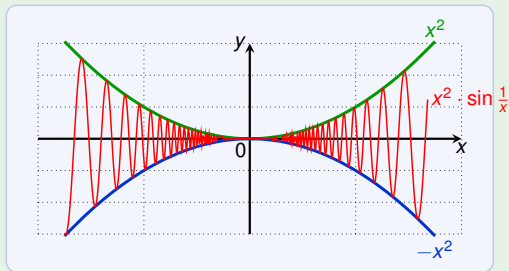


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Hence by the squeeze theorem we get:  $\lim_{x \rightarrow 0} x^2 \cdot \sin \frac{1}{x} = 0$ .