

Calculus M211

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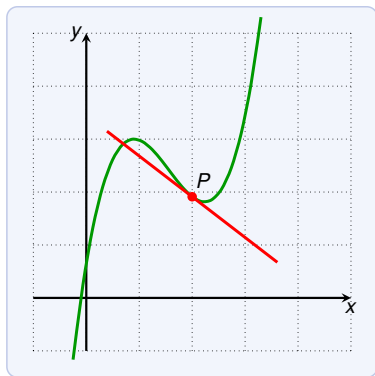
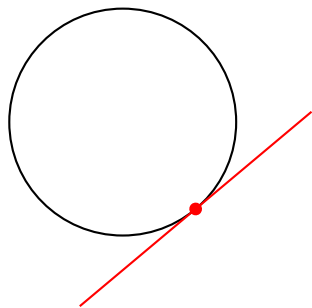
Indiana University Bloomington

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The Tangent

The **tangent** is a line that touches the curve:

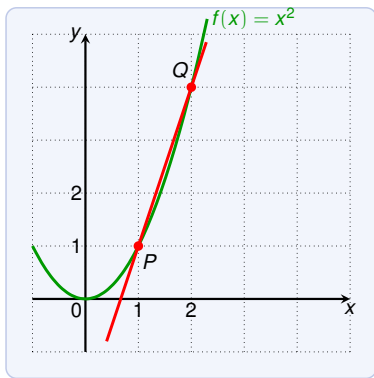
- ▶ same direction as the curve at the point of contact



The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.

- ▶ We need to know the slope m of x^2 at point $P = (1, 1)$.
- ▶ Take point $Q = (x, x^2)$ with $Q \neq P$ to compute the slope.



The slope from P to Q is:

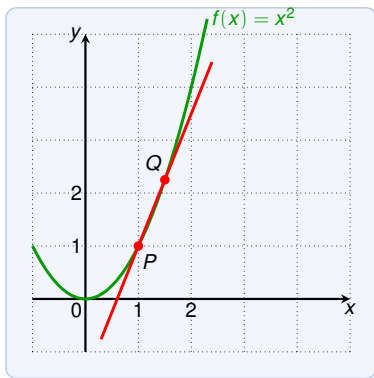
$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

| | | | | |
|----------|---|--|--|--|
| x | 2 | | | |
| m_{PQ} | 3 | | | |

The Tangent: Example

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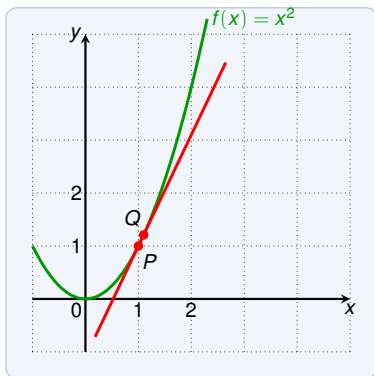
$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

| | | | | |
|----------|---|-----|--|--|
| x | 2 | 1.5 | | |
| m_{PQ} | 3 | 2.5 | | |

The Tangent: Example

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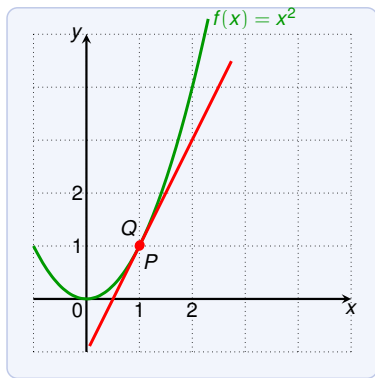
$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

| | | | | |
|----------|---|-----|-----|--|
| x | 2 | 1.5 | 1.1 | |
| m_{PQ} | 3 | 2.5 | 2.1 | |

The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.

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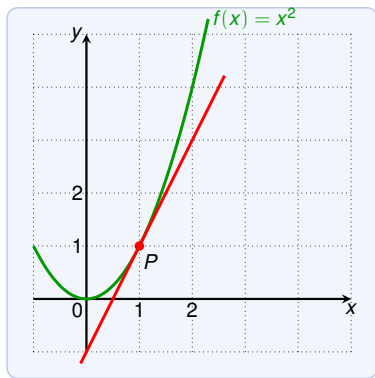
| | | | | | |
|----------|---|-----|-----|------|-----|
| x | 2 | 1.5 | 1.1 | 1.01 | ... |
| m_{PQ} | 3 | 2.5 | 2.1 | 2.01 | ... |

The closer Q to P , the closer m_{PQ} gets to 2.

The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.

- ▶ We need to know the slope m of x^2 at point $P = (1, 1)$.
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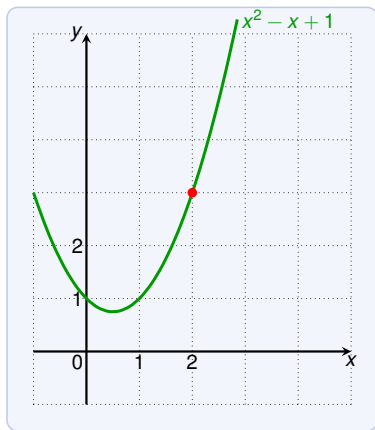
| | | | | | |
|----------|---|-----|-----|------|-----|
| x | 2 | 1.5 | 1.1 | 1.01 | ... |
| m_{PQ} | 3 | 2.5 | 2.1 | 2.01 | ... |

The closer Q to P , the closer m_{PQ} gets to 2. Suggests that in P the slope $m = 2$.

Thus the tangent is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

The Limit of a Function

We investigate the function $x^2 - x + 1$ for values of x near 2.



from below ($x < 2$):

| x | $f(x)$ |
|-------|--------|
| 1 | 1 |
| 1.5 | 1.75 |
| 1.9 | 2.71 |
| 1.99 | 2.9701 |
| 1.999 | 2.9970 |

from above ($x > 2$):

| x | $f(x)$ |
|-------|--------|
| 2.5 | 4.75 |
| 2.2 | 3.64 |
| 2.1 | 3.31 |
| 2.01 | 3.0301 |
| 2.001 | 3.0030 |

From the tables we see: as x approaches 2, $f(x)$ approaches 3.

$$\lim_{x \rightarrow 2} (x^2 - x + 1) = 3$$

Limit: Definition

Suppose $f(x)$ is defined close to a (but not necessarily a itself). We write

$$\lim_{x \rightarrow a} f(x) = L$$

spoken: “the limit of $f(x)$, as x approaches a , is L ”

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a .

The values of $f(x)$ get closer to L as x gets closer to a .

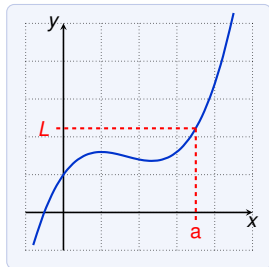
Alternative notation for $\lim_{x \rightarrow a} f(x) = L$:

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow a$$

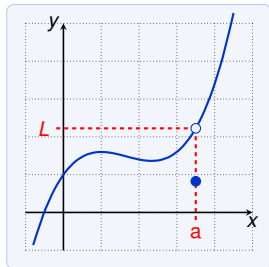
Limit: Continued

$\lim_{x \rightarrow a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a but not equal to a .

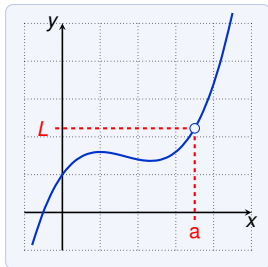
Note that we never consider $f(x)$ for $x = a$. **The value of $f(a)$ does not matter.** In fact, $f(x)$ need not be defined for $x = a$.



$$f(a) = L$$



$$f(a) \neq L$$



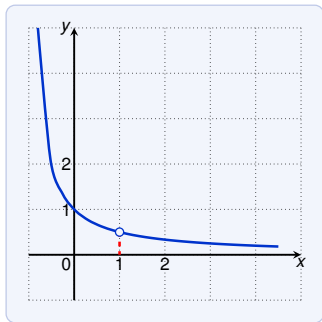
$f(a)$ undefined

In each of these cases we have $\lim_{x \rightarrow a} f(x) = L$!

Limit: Examples

Guess the value of

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$$



The function is not defined at $x = 1$.
(does not matter for the limit)

from below:

| x | $f(x)$ |
|-------|---------|
| 0.5 | 0.66667 |
| 0.9 | 0.52632 |
| 0.99 | 0.50251 |
| 0.999 | 0.50025 |

from above:

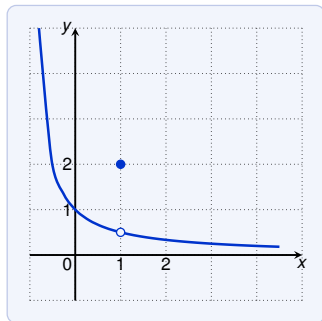
| x | $f(x)$ |
|-------|---------|
| 1.5 | 0.40000 |
| 1.1 | 0.47619 |
| 1.01 | 0.49751 |
| 1.001 | 0.49975 |

From these values we guess that $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5$.

Limit: Examples

Guess the value of $\lim_{x \rightarrow 1} g(x)$ where

$$g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$$



As on the previous slide $\lim_{x \rightarrow 1} g(x) = 0.5$.
(recall that $g(1)$ does not matter for $\lim_{x \rightarrow 1} g(x)$).

Limit: Examples

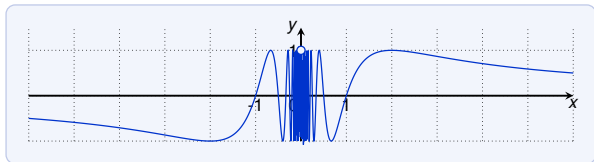
Guess the value of

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$$

| x | $f(x)$ |
|-------------|--------|
| ± 1 | 0 |
| ± 0.1 | 0 |
| ± 0.01 | 0 |
| ± 0.001 | 0 |

This suggest that the limit is 0.

However, this is **wrong**:



$\sin(\frac{\pi}{x}) = 0$ for arbitrarily small x , but also

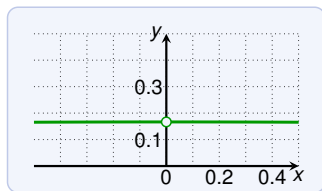
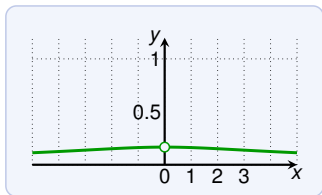
$\sin(\frac{\pi}{x}) = 1$ for arbitrarily small x ; e.g. $x = \frac{1}{2.5}, \frac{1}{4.5}, \frac{1}{6.5}, \dots$

Hence: **The limit $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$ does not exist.**

Limit: Caution with Calculators

Guess the value of

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$



| x | $f(x)$ |
|----------------|---------|
| ± 1.0 | 0.16228 |
| ± 0.5 | 0.16553 |
| ± 0.1 | 0.16662 |
| ± 0.01 | 0.16667 |
| ± 0.0001 | 0.20000 |
| ± 0.00001 | 0.00000 |
| ± 0.000001 | 0.00000 |

Is the limit 0? **NO**

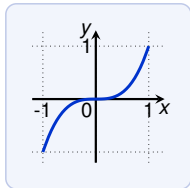
Problem: calculator gives wrong values!
For small x it rounds $\sqrt{x^2 + 9} - 3$ to 0.

The correct limit is $\frac{1}{6} = 0.166666\dots$

Limit: Examples

Guess the value of

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10000} \right)$$



| x | $f(x)$ |
|------|----------|
| 1 | 1.000028 |
| 0.5 | 0.124920 |
| 0.1 | 0.001088 |
| 0.01 | 0.000101 |

Looks like the limit is 0. But if we continue:

| x | $f(x)$ |
|-------|------------|
| 0.005 | 0.00010009 |
| 0.001 | 0.00010000 |

We see actually that:

The value of the limit is 0.0001.

Limits and Calculators

Determining limits via calculators is a bad idea!

We have seen several sources of errors:

- ▶ we might stop too early, and draw wrong conclusions
- ▶ wrong results due to rounding in the calculator

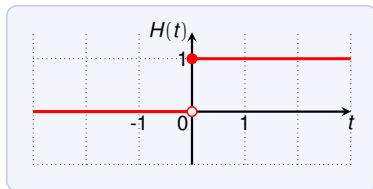
We need to compute limits precisely using limits laws. . .

Limit: Examples

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

What is $\lim_{t \rightarrow 0} H(t)$?



- ▶ As t approaches 0 from the left, $H(t)$ approaches 0.
- ▶ As t approaches 0 from the right, $H(t)$ approaches 1.

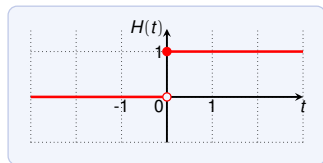
Thus there is not single number that $H(t)$ approaches.

The limit $\lim_{t \rightarrow 0} H(t)$ does not exist.

One-Sided Limits (From the Left)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$H(t)$ approaches 0 as t approaches 0 from the **left**. We write:

$$\lim_{t \rightarrow 0^-} H(t) = 0$$

The symbol $t \rightarrow 0^-$ indicates that we consider only values $t < 0$.

We write $\lim_{x \rightarrow a^-} f(x) = L$ and say

“the **left-hand** limit of $f(x)$, as x approaches a , is L ”, or

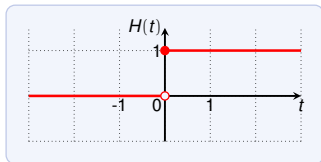
“the limit of $f(x)$, as x approaches a **from the left**, is L ”

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

One-Sided Limits (From the Right)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$H(t)$ approaches 1 as t approaches 0 from the **right**. We write:

$$\lim_{t \rightarrow 0^+} H(t) = 1$$

The symbol $t \rightarrow 0^+$ indicates that we consider only values $t > 0$.

We write $\lim_{x \rightarrow a^+} f(x) = L$ and say

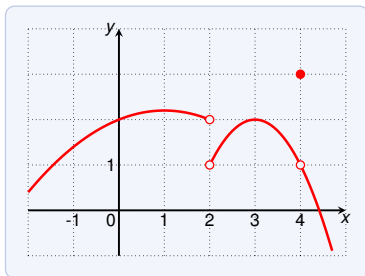
“the **right-hand** limit of $f(x)$, as x approaches a , is L ”, or

“the limit of $f(x)$, as x approaches a **from the right**, is L ”

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

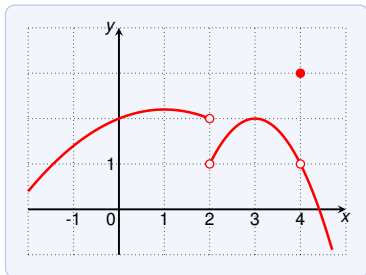


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

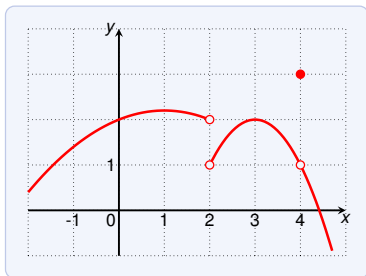


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = ?$ **a** 0 **b** 1 **c** 2 **d** 3 **e** does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

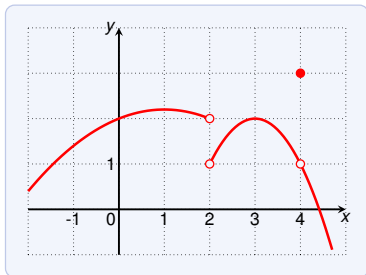


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = 1$
- ▶ $\lim_{x \rightarrow 2} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

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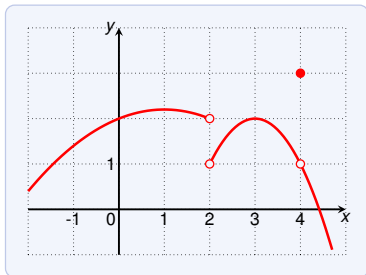


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = 1$
- ▶ $\lim_{x \rightarrow 2}$ does not exist
- ▶ $\lim_{x \rightarrow 4^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

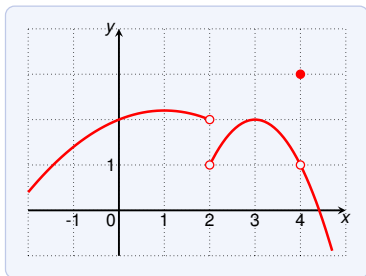


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = 1$
- ▶ $\lim_{x \rightarrow 2}$ does not exist
- ▶ $\lim_{x \rightarrow 4^-} = 1$
- ▶ $\lim_{x \rightarrow 4^+} = ?$ **a** 0 **b** 1 **c** 2 **d** 3 **e** does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

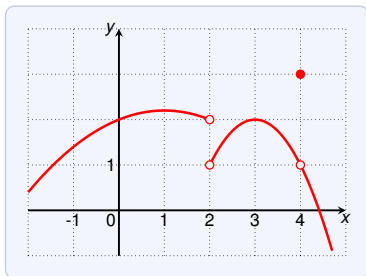


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = 1$
- ▶ $\lim_{x \rightarrow 2}$ does not exist
- ▶ $\lim_{x \rightarrow 4^-} = 1$
- ▶ $\lim_{x \rightarrow 4^+} = 1$
- ▶ $\lim_{x \rightarrow 4} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

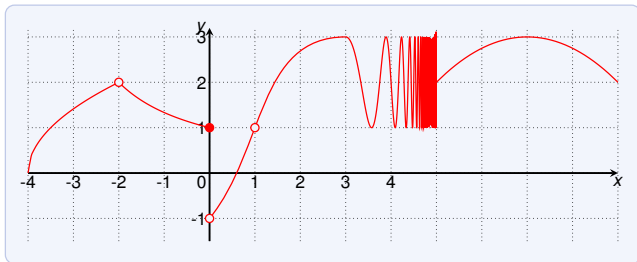


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = 1$
- ▶ $\lim_{x \rightarrow 2}$ does not exist
- ▶ $\lim_{x \rightarrow 4^-} = 1$
- ▶ $\lim_{x \rightarrow 4^+} = 1$
- ▶ $\lim_{x \rightarrow 4} = 1$

Infinite Limits: Example

Consider the following graph of function $g(x)$:

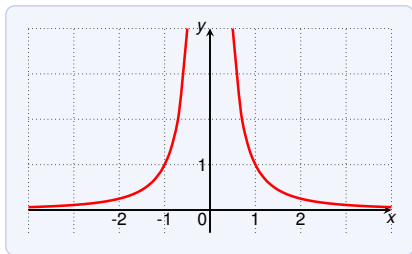


Use the graph to estimate the following values:

- ▶ $\lim_{x \rightarrow 3^-} g(x) = 3$
- ▶ $\lim_{x \rightarrow 3^+} g(x) = 3$
- ▶ $\lim_{x \rightarrow 3} g(x) = 3$
- ▶ $\lim_{x \rightarrow 1} g(x) = 1$
- ▶ $g(1) = \text{undefined}$
- ▶ $g(0) = 1$
- ▶ $\lim_{x \rightarrow 0^-} g(x) = 1$
- ▶ $\lim_{x \rightarrow 0^+} g(x) = -1$
- ▶ $\lim_{x \rightarrow 0} g(x) = \text{does not exist}$
- ▶ $\lim_{x \rightarrow 5^-} g(x) = \text{does not exist}$
- ▶ $\lim_{x \rightarrow 5^+} g(x) = 2$
- ▶ $\lim_{x \rightarrow 5} g(x) = \text{does not exist}$

Infinite Limits

We consider the function $\frac{1}{x^2}$. What is $\lim_{x \rightarrow 0} \frac{1}{x^2}$?



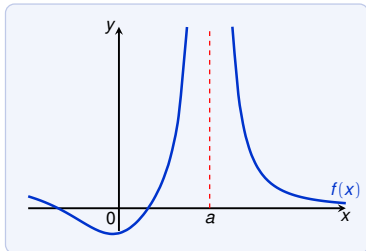
As x becomes close to 0, $\frac{1}{x^2}$ becomes very large. The values do not approach a number, so $\lim_{x \rightarrow 0} \frac{1}{x^2}$ **does not exist!**

Nevertheless, in this case, we write

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

to indicate that the values become larger and larger.

Infinite Limits: Definition



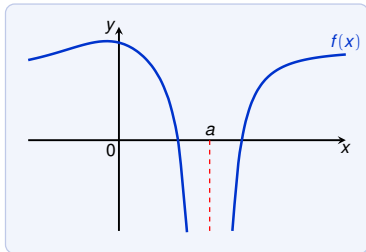
Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \rightarrow a} f(x) = \infty$$

spoken: “the limit of $f(x)$, as x approaches a , is **infinity**”

if we can make the values of $f(x)$ **arbitrarily large** by taking x to be sufficiently close to a (but not equal to a).

Infinite Limits: Definition



Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

spoken: “the limit of $f(x)$, as x approaches a , is **negative infinity**”

if we can make the values of $f(x)$ **arbitrarily large negative** by taking x to be sufficiently close to a (but not equal to a).

Infinite One-Sided Limits

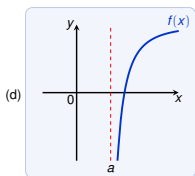
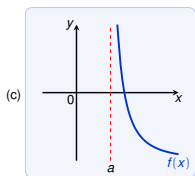
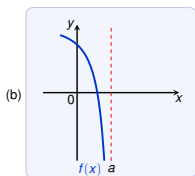
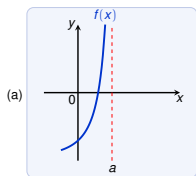
Like wise we define the one-sided infinite limits:

(a) $\lim_{x \rightarrow a^-} f(x) = \infty$

(b) $\lim_{x \rightarrow a^-} f(x) = -\infty$

(c) $\lim_{x \rightarrow a^+} f(x) = \infty$

(d) $\lim_{x \rightarrow a^+} f(x) = -\infty$



Note that ∞ and $-\infty$ are not considered numbers.

If $\lim_{x \rightarrow a} f(x) = \infty$ then $\lim_{x \rightarrow a} f(x)$ does not exist.

It indicates a certain way in which the limit does not exist.

Infinite Limits: Examples

Find

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$$

and

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$$

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = ? \quad \text{a } 0 \quad \text{b } 1 \quad \text{c } \infty \quad \text{d } -\infty$$

Infinite Limits: Examples

Find

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$$

and

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$$

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty$$

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = ? \quad \text{a } 0 \quad \text{b } 1 \quad \text{c } \infty \quad \text{d } -\infty$$

Infinite Limits: Examples

Find

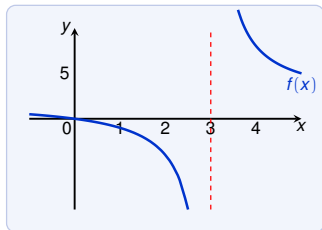
$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$$

and

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$$

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty$$

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty$$



If x is close to 3 and $x < 3$ (approaching from the left), then:

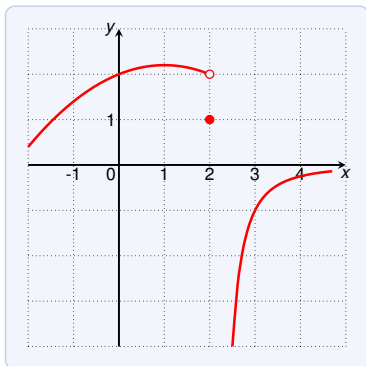
- ▶ $2x$ is close to 6,
- ▶ $x - 3$ is a small negative number,
- ▶ and thus $2x/(x - 3)$ is a large negative number.

Hence $\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty$.

Similarly for x close to 3 and $x > 3$, but now $x - 3$ is positive.

Infinite Limits: Example

Consider the following graph of function $g(x)$:

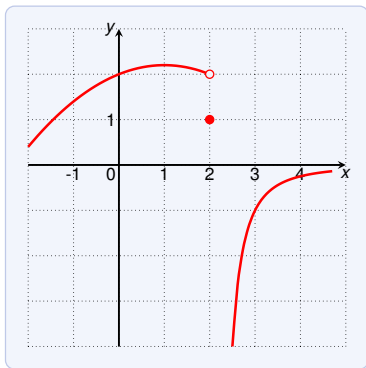


Use the graph to estimate the following values:

- $f(2) = ?$ **a** 0 **b** 1 **c** 2 **d** undefined

Infinite Limits: Example

Consider the following graph of function $g(x)$:

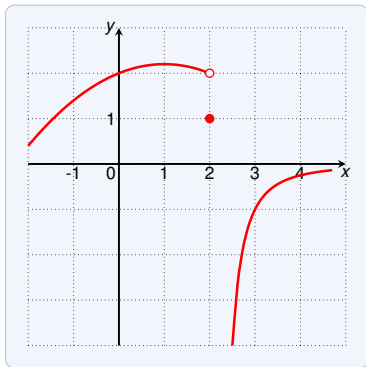


Use the graph to estimate the following values:

- ▶ $f(2) = 1$
- ▶ $\lim_{x \rightarrow 2^-} = ?$ **a** 1 **b** 2 **c** ∞ **d** $-\infty$ **e** does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:

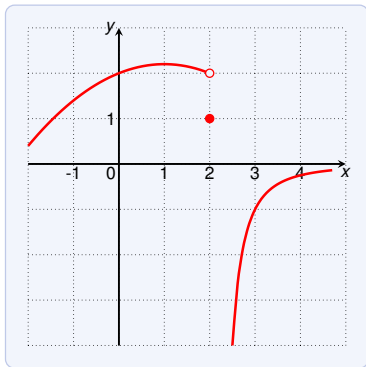


Use the graph to estimate the following values:

- ▶ $f(2) = 1$
- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = ?$ **a** 1 **b** 2 **c** ∞ **d** $-\infty$ **e** does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:

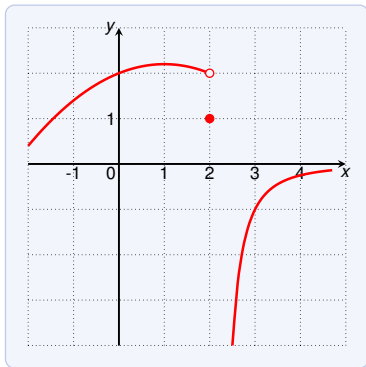


Use the graph to estimate the following values:

- ▶ $f(2) = 1$
- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = -\infty$ (special case of 'does not exist')
- ▶ $\lim_{x \rightarrow 2} = ?$ **a** 1 **b** ∞ **c** $-\infty$ **d** does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

- ▶ $f(2) = 1$
- ▶ $\lim_{x \rightarrow 2^-} = 2$
- ▶ $\lim_{x \rightarrow 2^+} = -\infty$ (special case of 'does not exist')
- ▶ $\lim_{x \rightarrow 2}$ does not exist

Infinite Limits: Vertical Asymptotes

The line $x = a$ is a **vertical asymptote** of a function f if at least one of the following statements is true:

$$\lim_{x \rightarrow a} f(x) = \infty$$

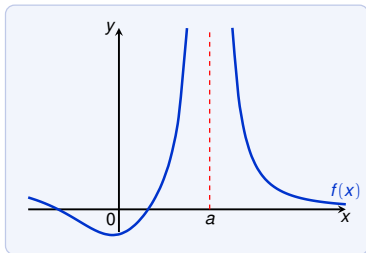
$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

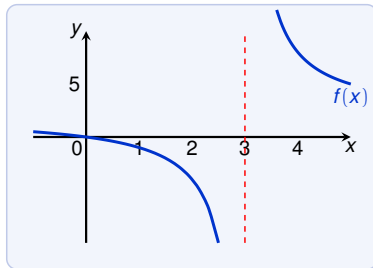
$$\lim_{x \rightarrow a^+} f(x) = -\infty$$



Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \frac{2x}{x-3} ?$$



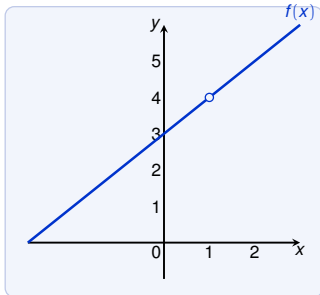
The function has the vertical asymptote $x = 3$:

$$\lim_{x \rightarrow 3^-} f(x) = -\infty$$

Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} ?$$



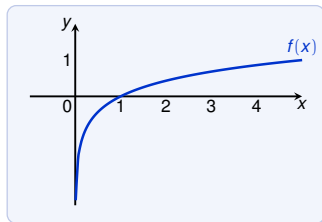
The function has no vertical asymptotes:

$$\frac{x^2 + 2x - 3}{x - 1} = (x + 3) \text{ for } x \neq 1$$

Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \log_5 x \text{ ?}$$



The function has the vertical asymptote $x = 0$:

$$\lim_{x \rightarrow 0^+} \log_5 x = -\infty$$