Calculus M211

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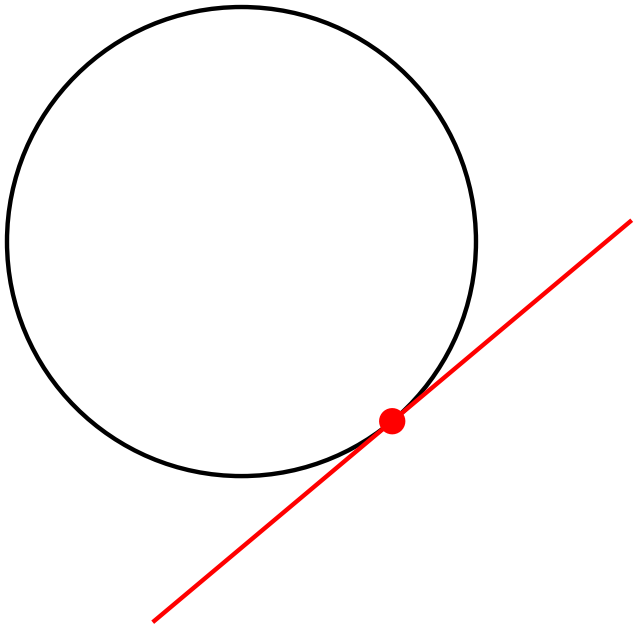
Indiana University Bloomington

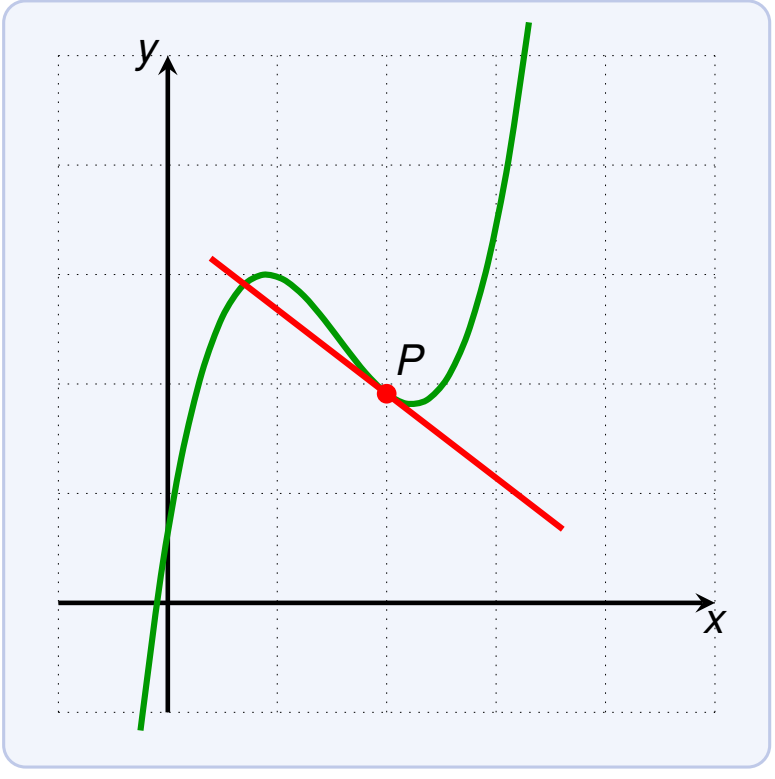
2013

The Tangent

The **tangent** is a line that touches the curve:

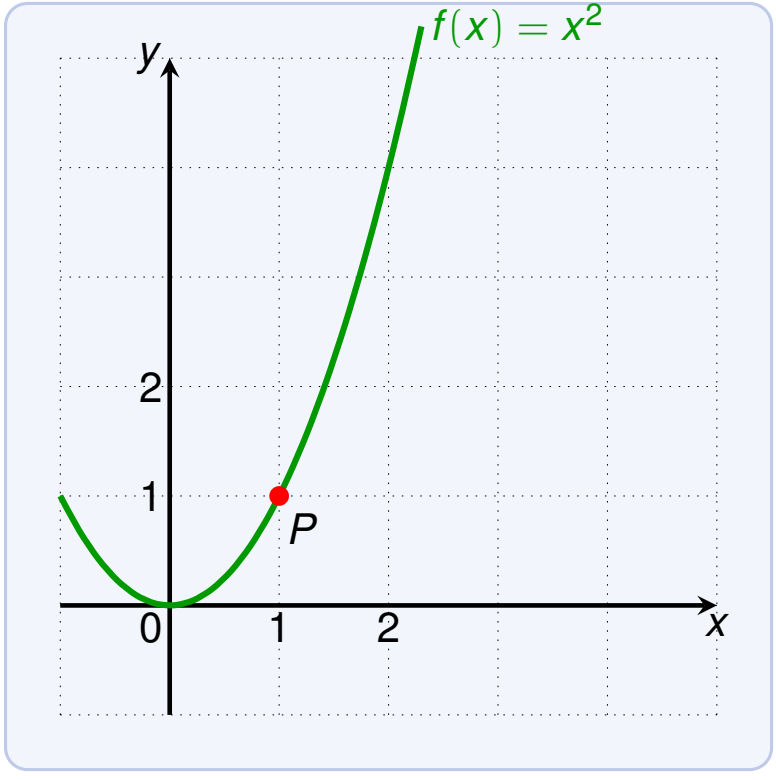
- same direction as the curve at the point of contact





The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.



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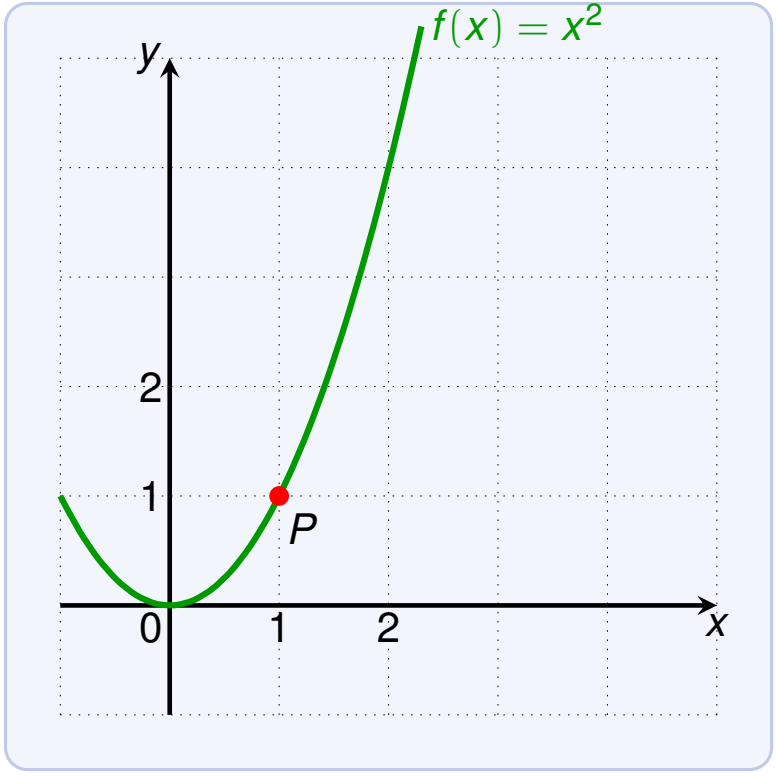

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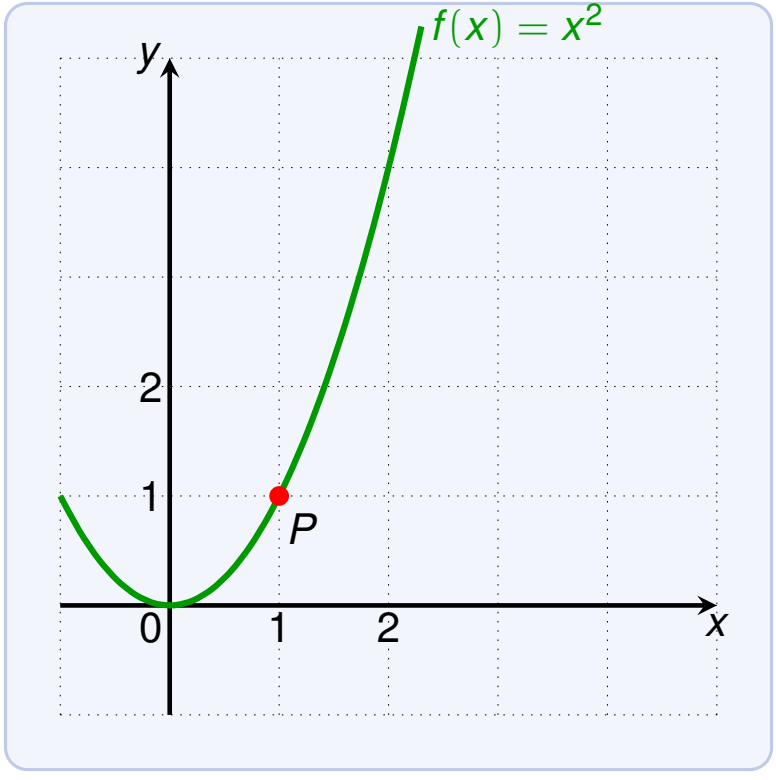

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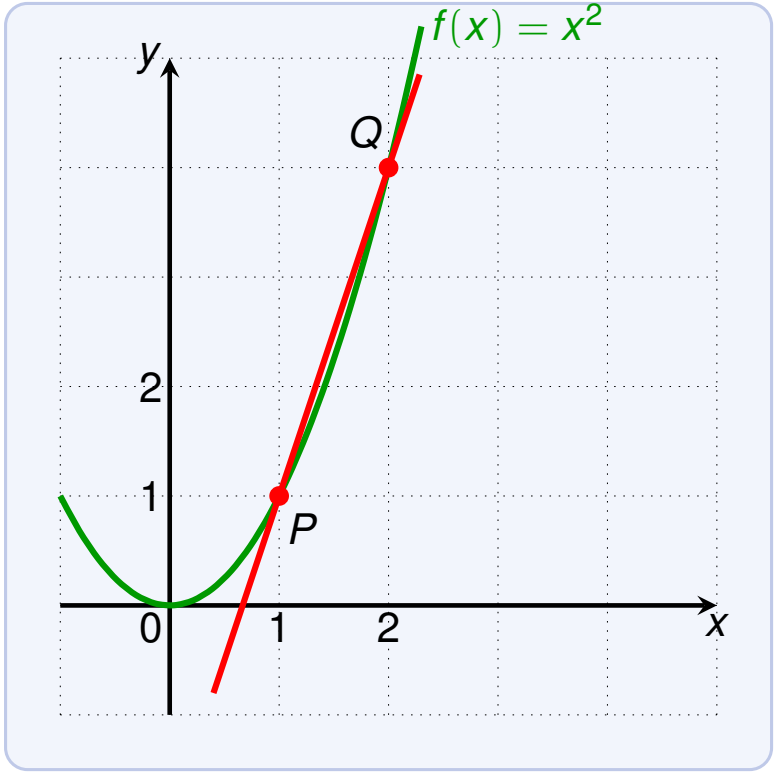

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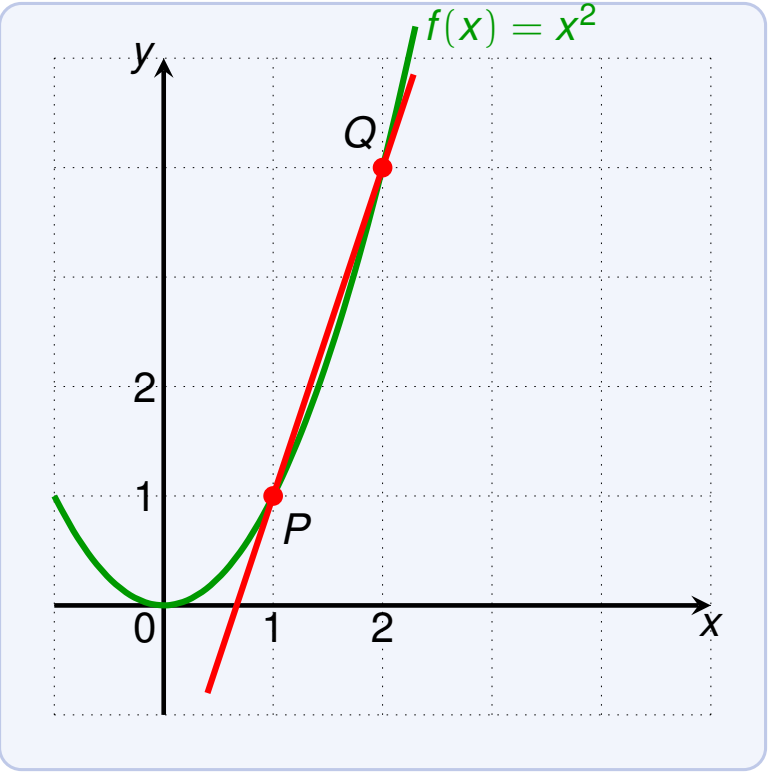

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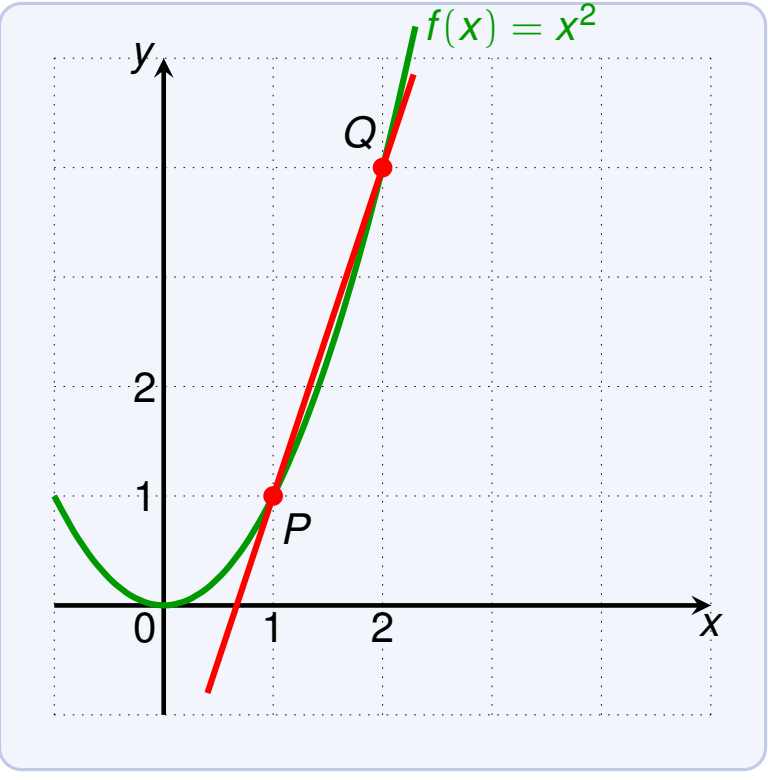

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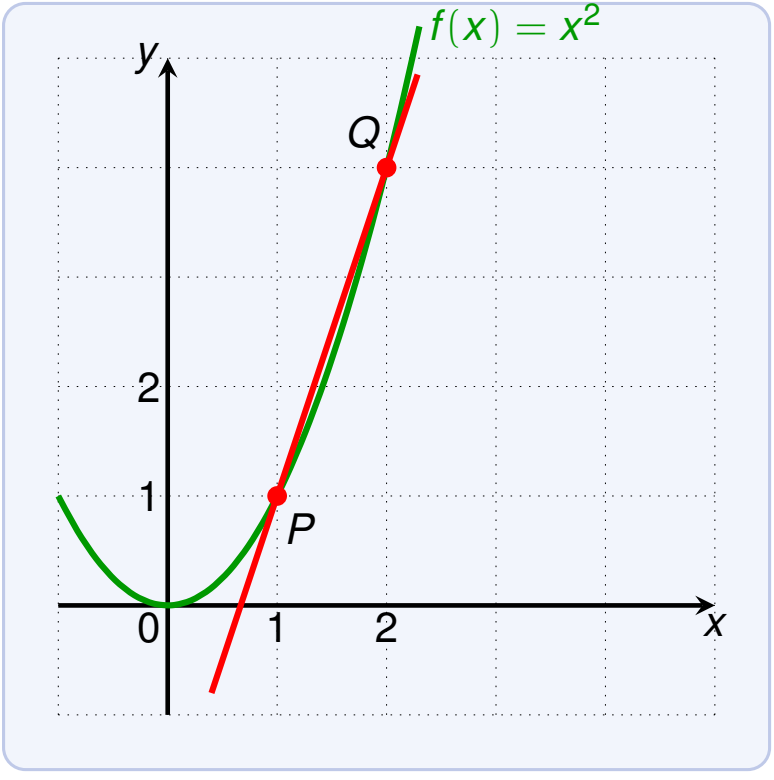

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m_{PQ}				

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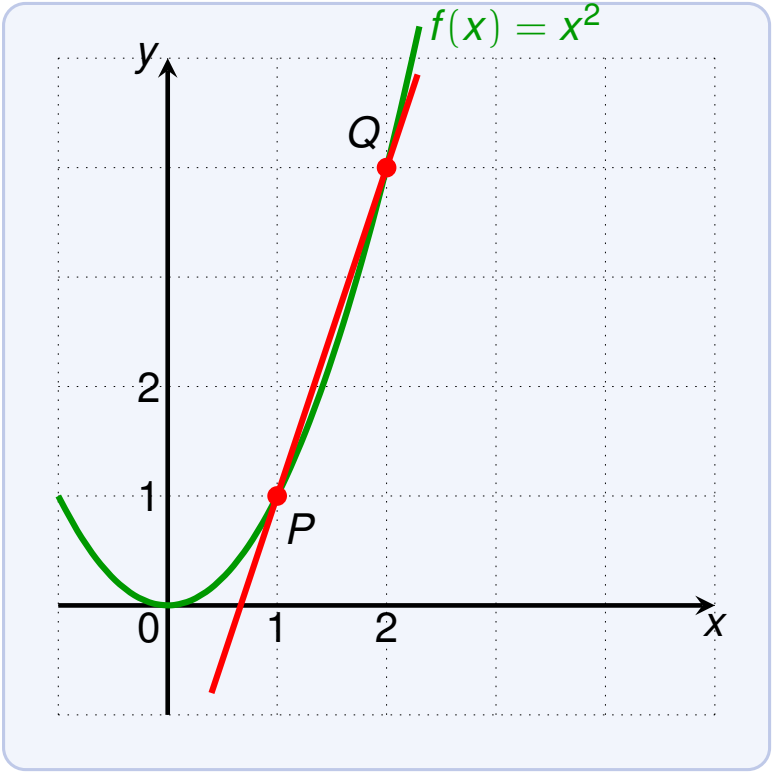

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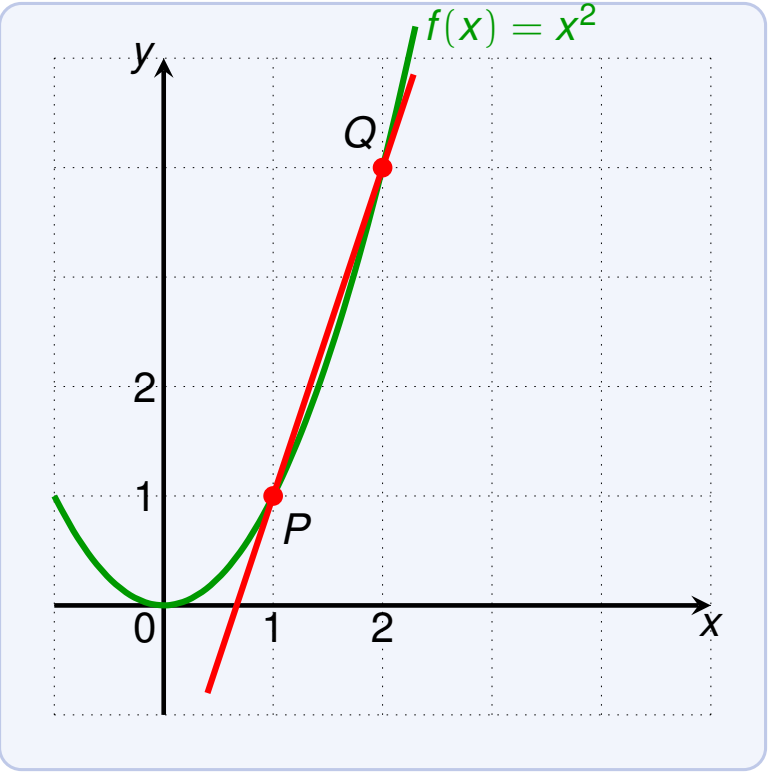

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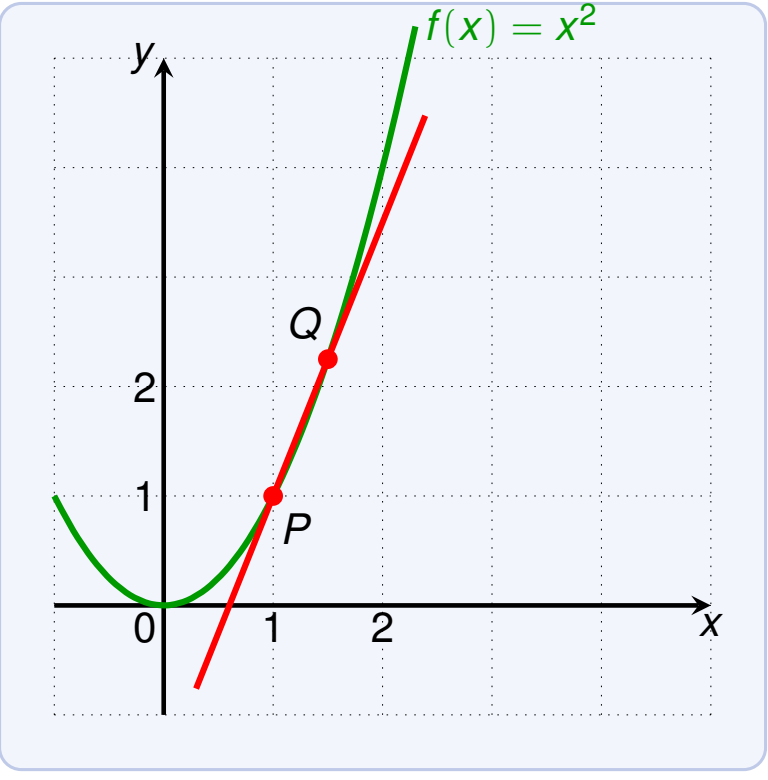

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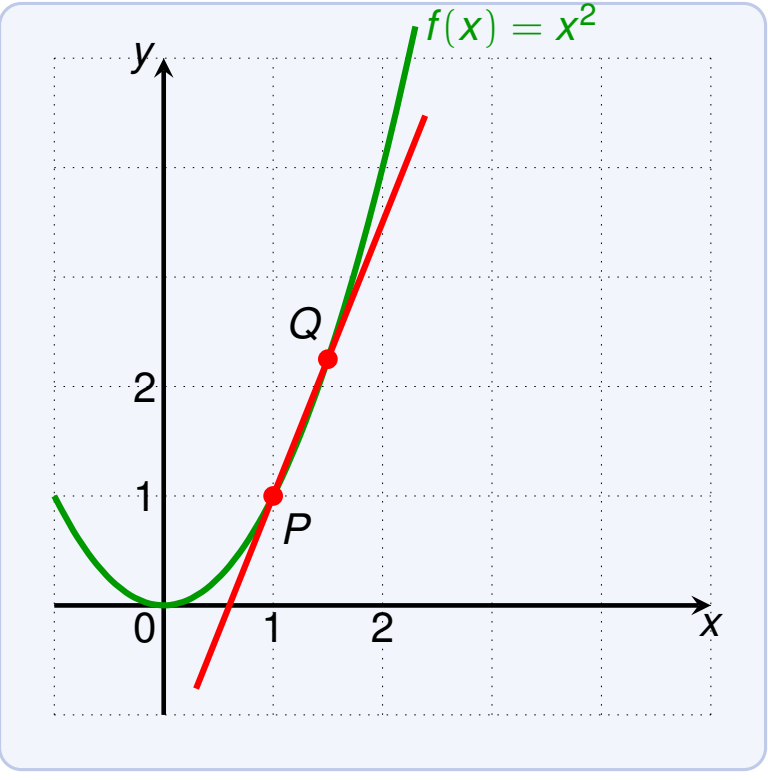

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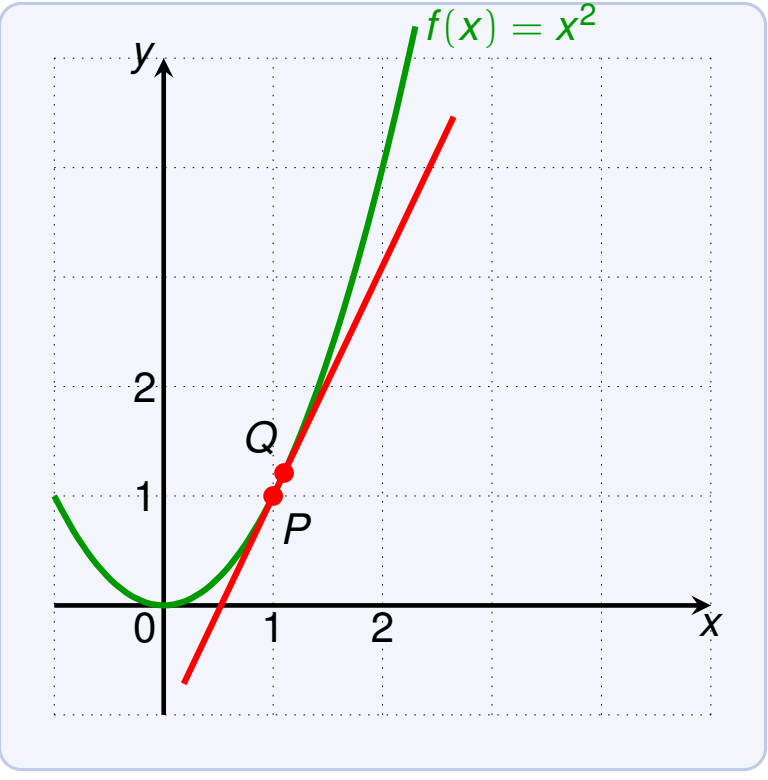

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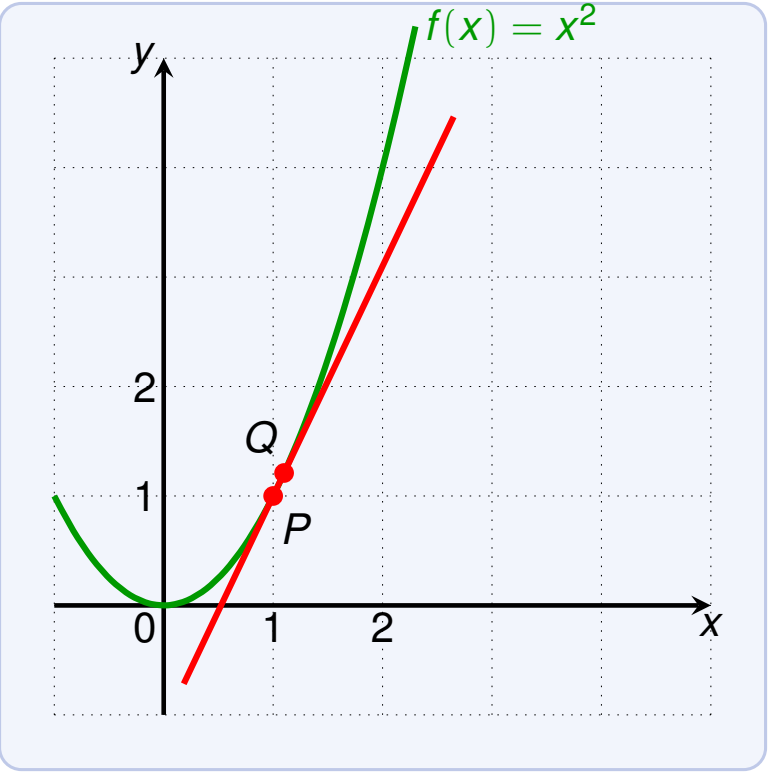

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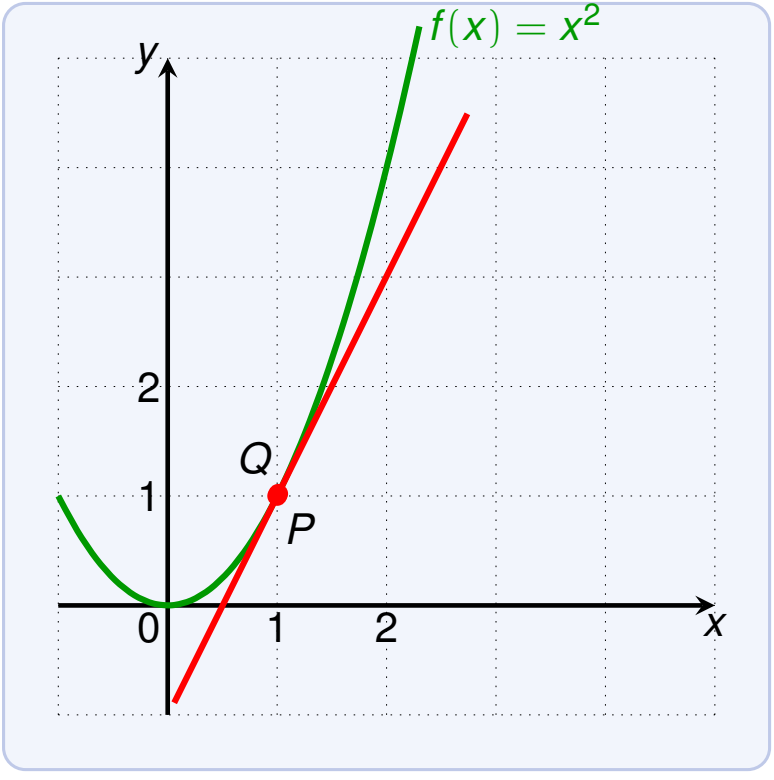

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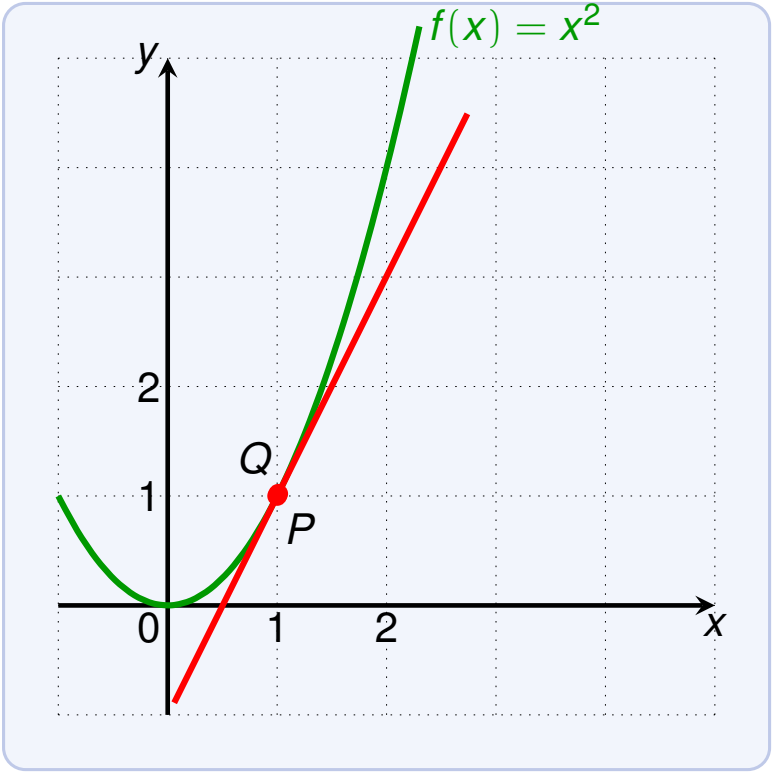

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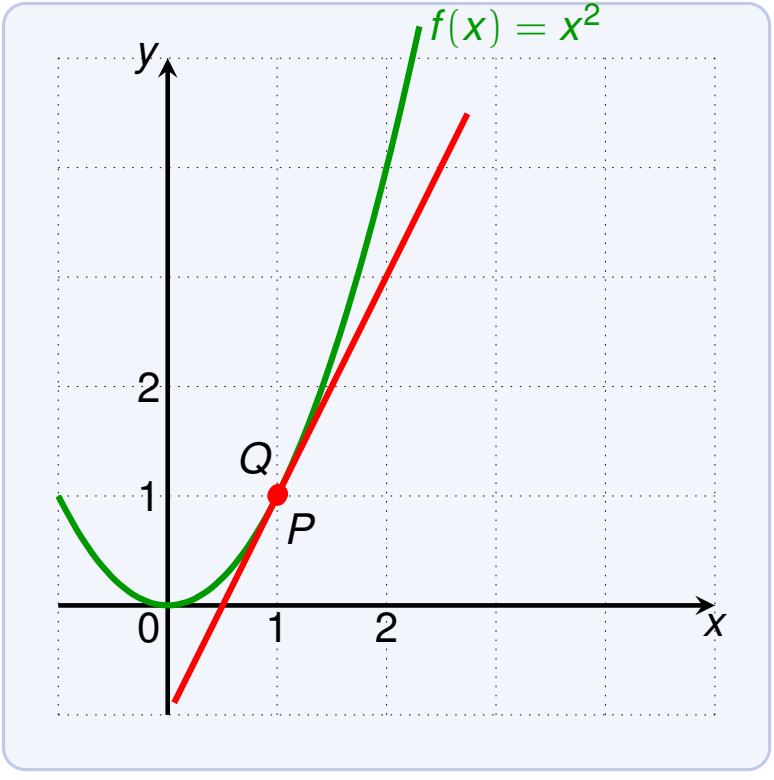

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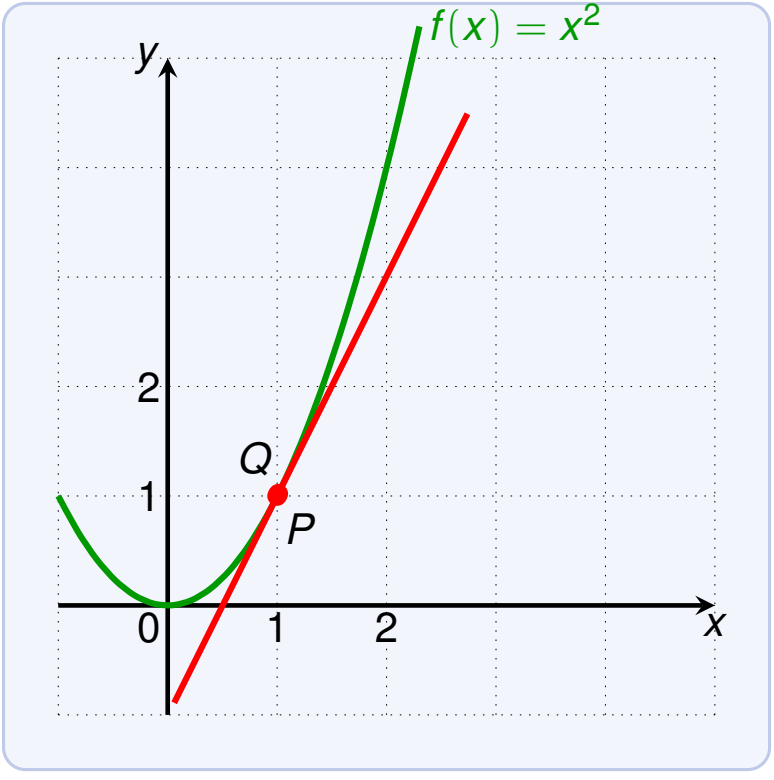

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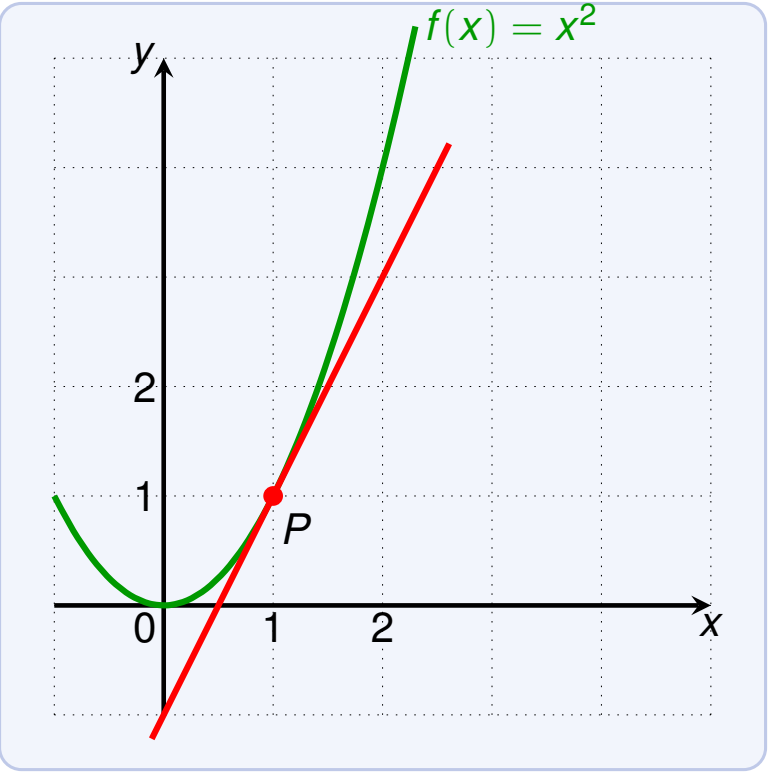

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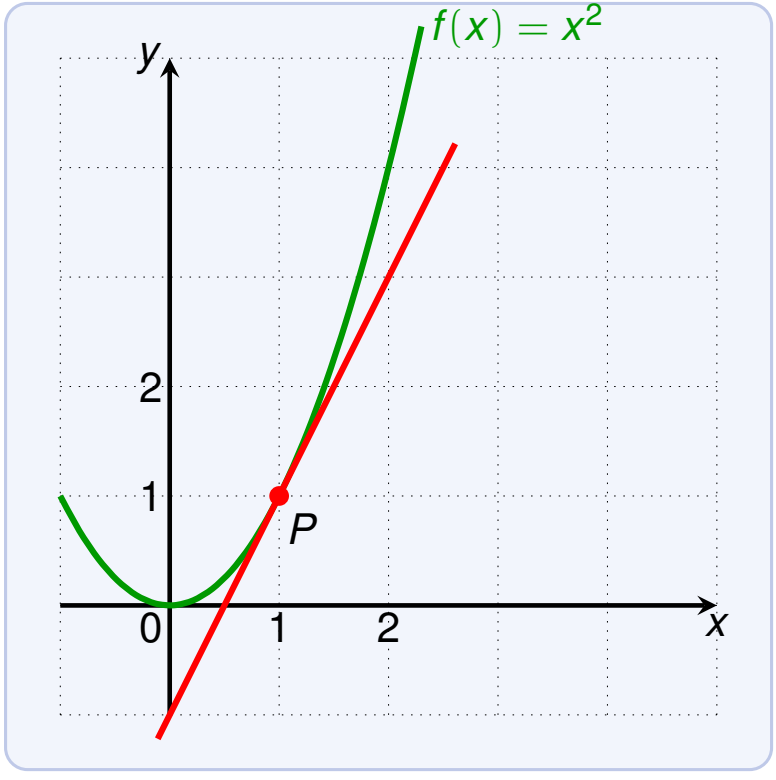

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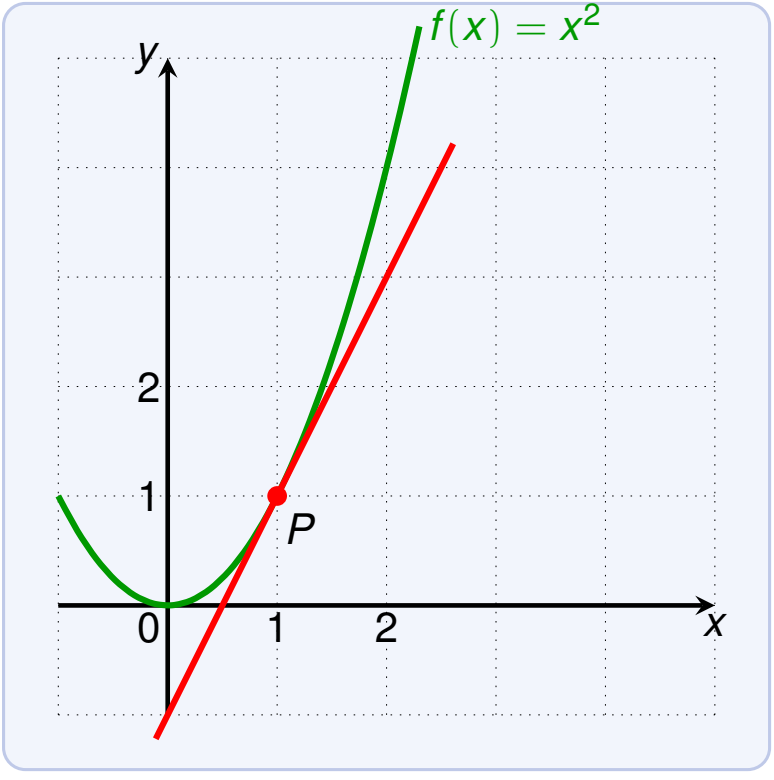

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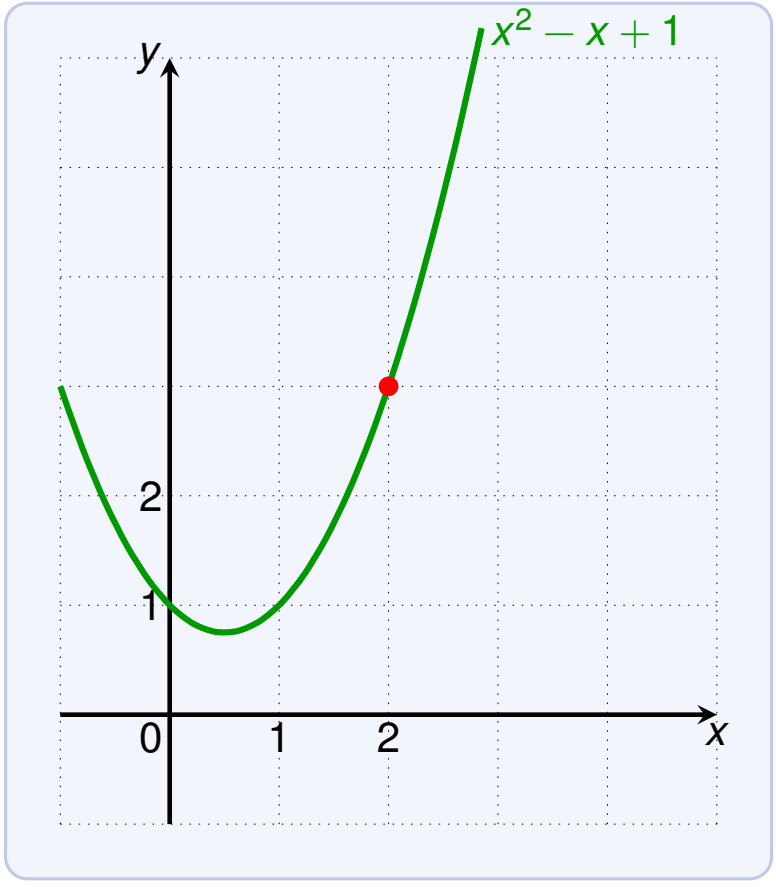
Thus the tangent is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

The Limit of a Function

We investigate the function $x^2 - x + 1$ for values of x near 2.



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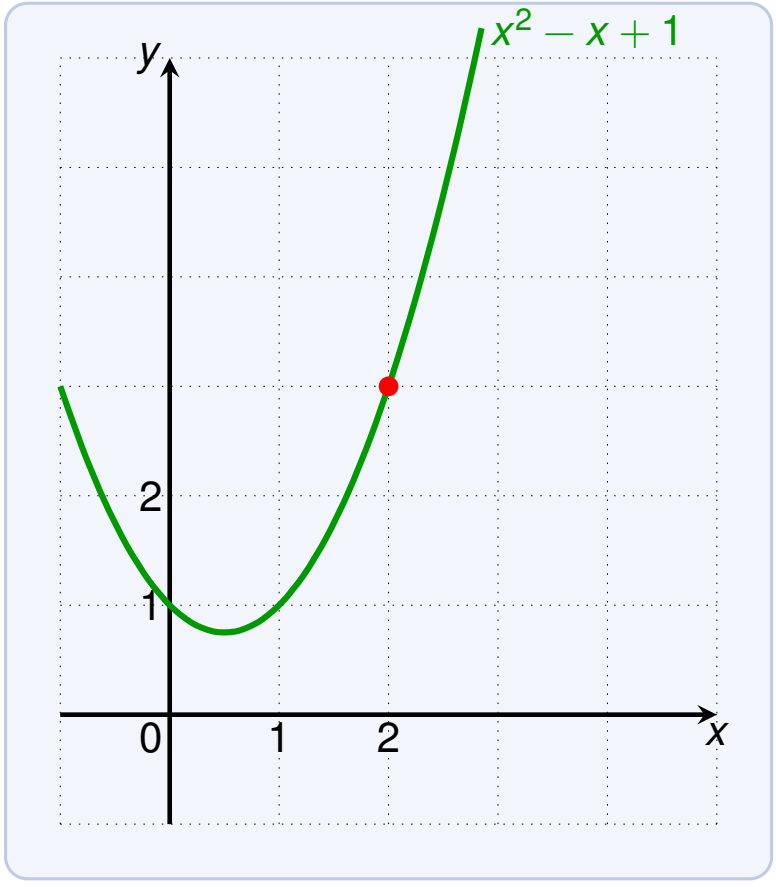

from below ($x < 2$):

x	$f(x)$
1	1
1.5	1.75
1.9	2.71
1.99	2.9701
1.999	2.9970

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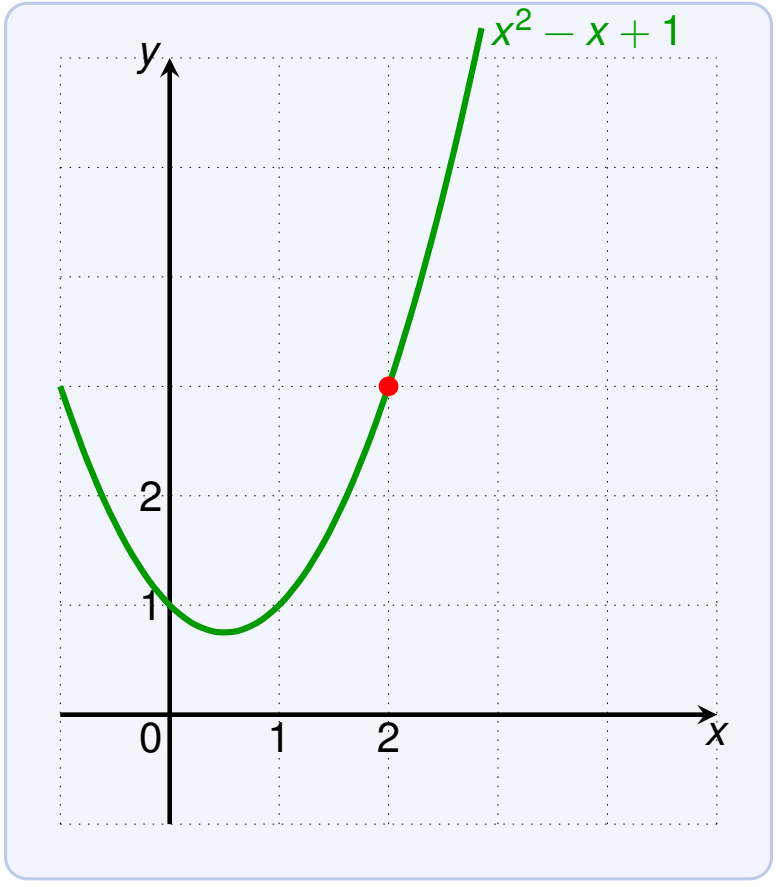

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from above ($x > 2$):

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2.2	3.64
2.1	3.31
2.01	3.0301
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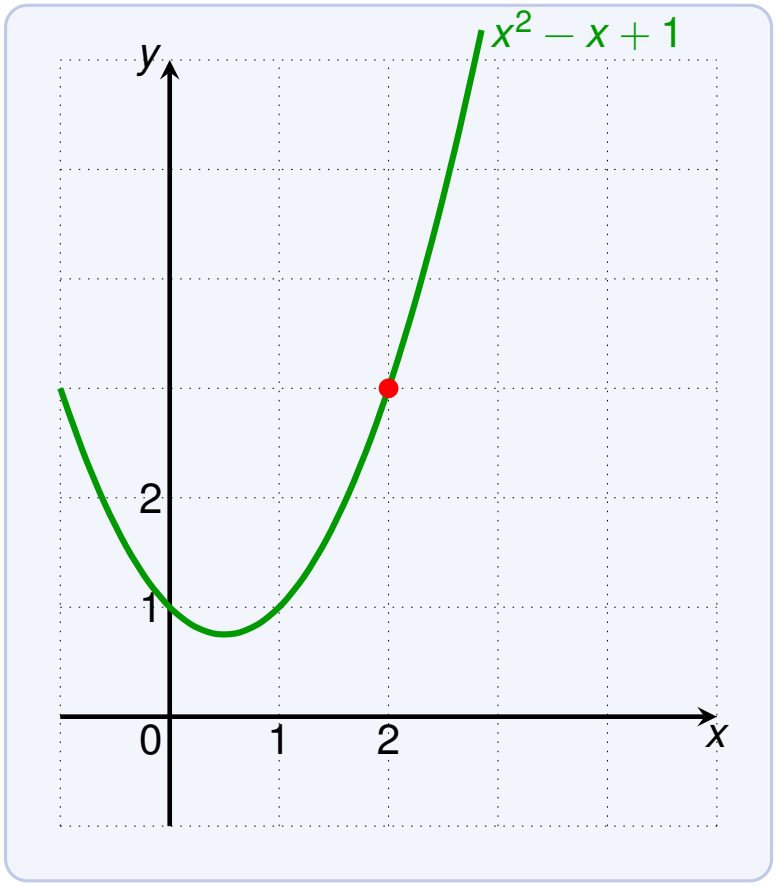

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x	$f(x)$
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2.01	3.0301
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From the tables we see: as x approaches 2, $f(x)$ approaches 3.

$$\lim_{x \to 2} (x^2 - x + 1) = 3$$

Limit: Definition

Suppose $f(x)$ is defined close to a (but not necessarily a itself). We write

$$\lim_{x \to a} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches a, is L"

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

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The values of $f(x)$ get closer to L as x gets closer to a.

Alternative notation for $\lim_{x \to a} = L$:

$$f(x) \to L \quad \text{as} \quad x \to a$$

Limit: Continued

$\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a but not equal to a.

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Note that we never consider $f(x)$ for $x = a$. The value of $f(a)$ does not matter. In fact, $f(x)$ need not be defined for $x = a$.

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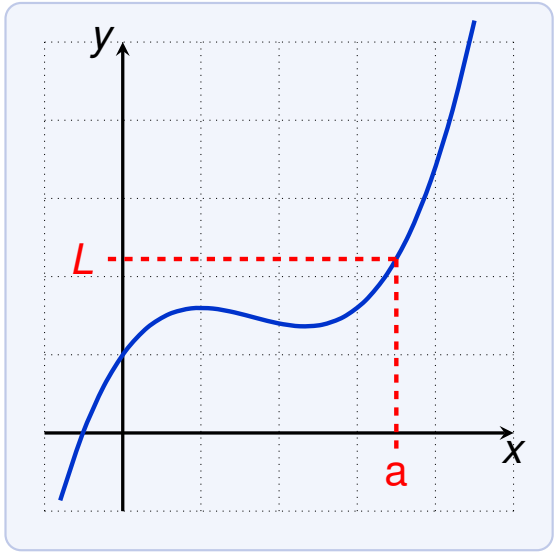
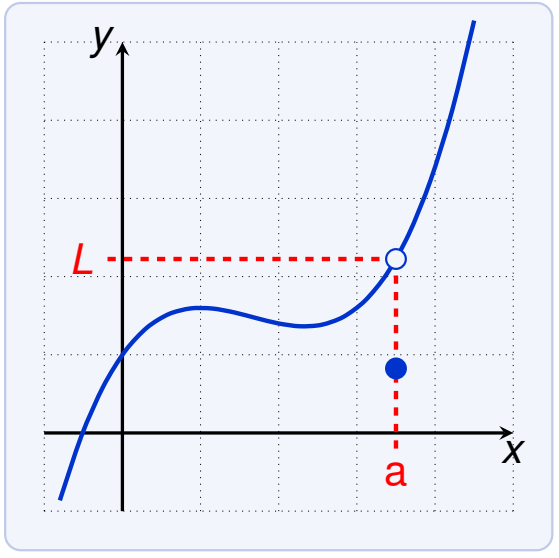
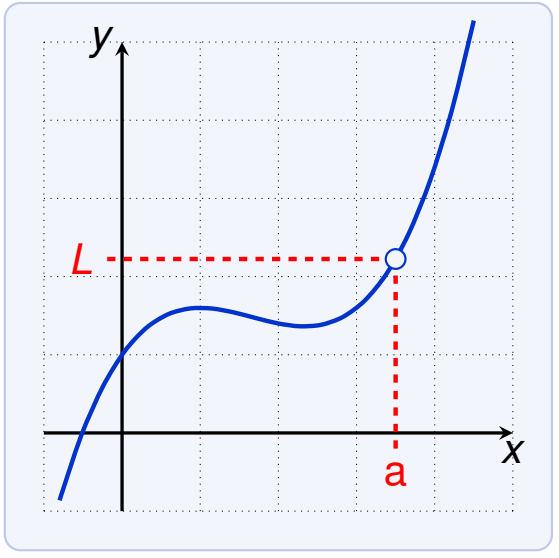


$f(a) = L$



$f(a) \neq L$

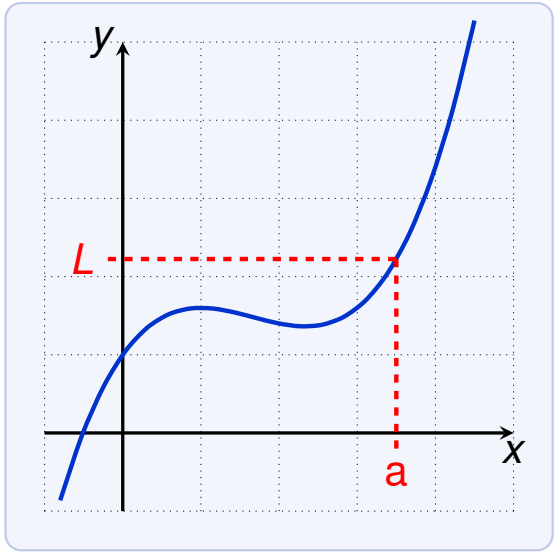


$f(a)$ undefined

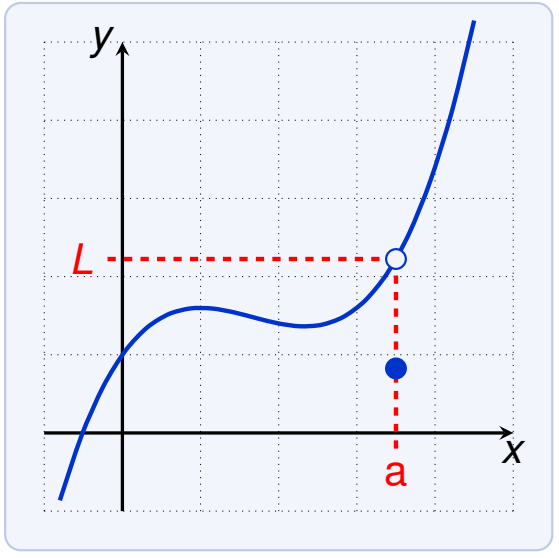
Limit: Continued

$\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a but not equal to a.

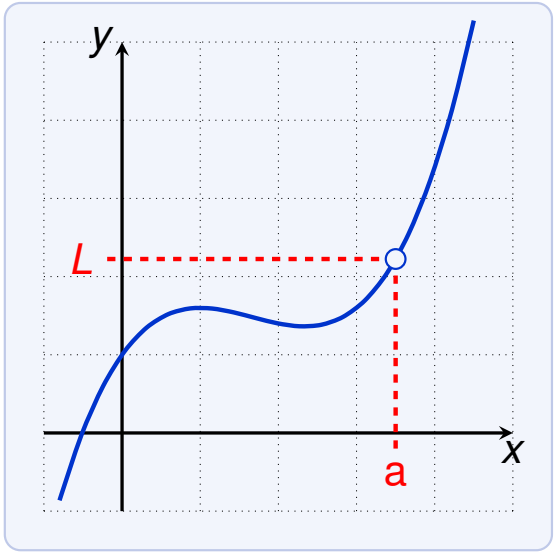
Note that we never consider $f(x)$ for $x = a$. The value of $f(a)$ does not matter. In fact, $f(x)$ need not be defined for $x = a$.



$f(a) = L$



$f(a) \neq L$



$f(a)$ undefined

In each of these cases we have $\lim_{x \to a} f(x) = L$!

Limit: Examples

Guess the value of

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$

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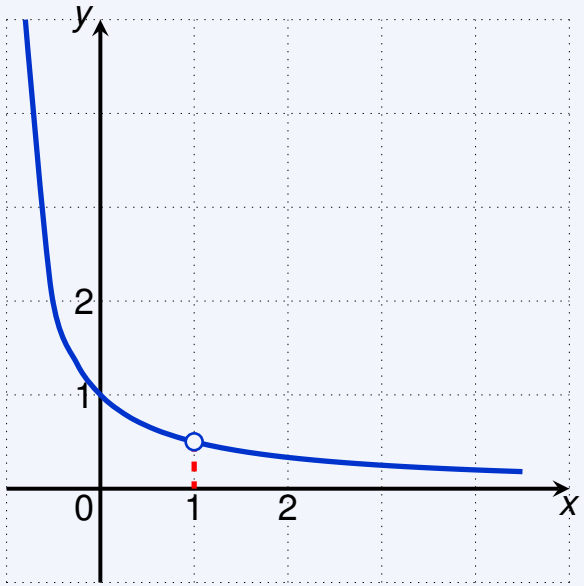

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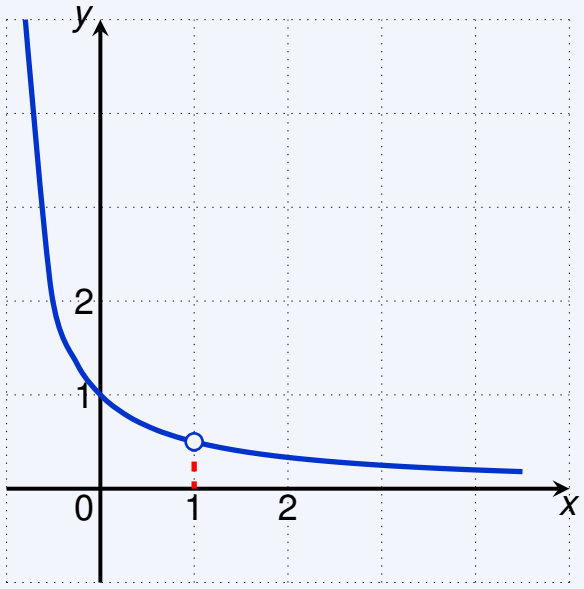

The function is not defined at $x = 1$.

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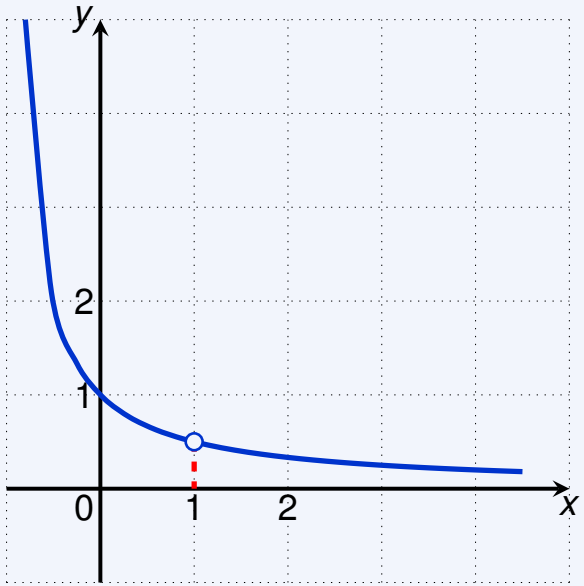

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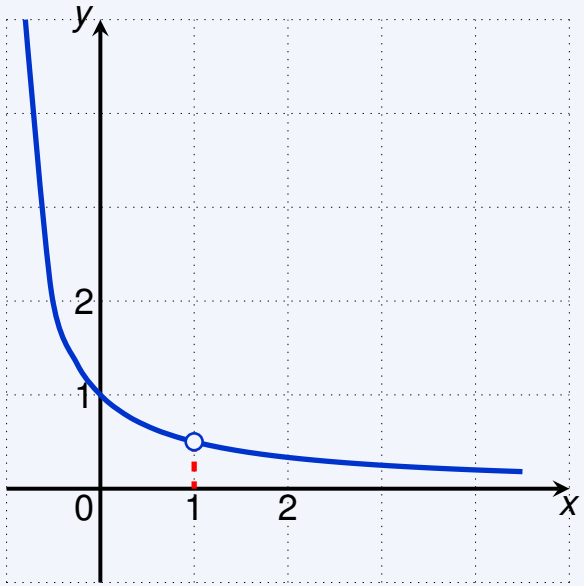

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from below:

x	$f(x)$
0.5	0.66667
0.9	0.52632
0.99	0.50251
0.999	0.50025

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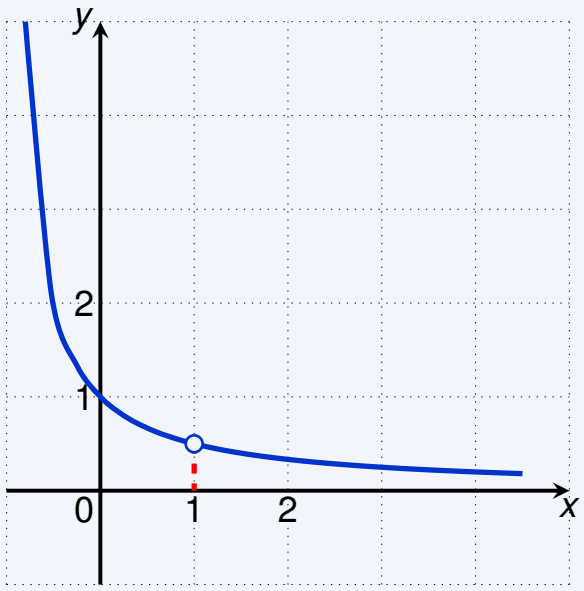

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1.1	0.47619
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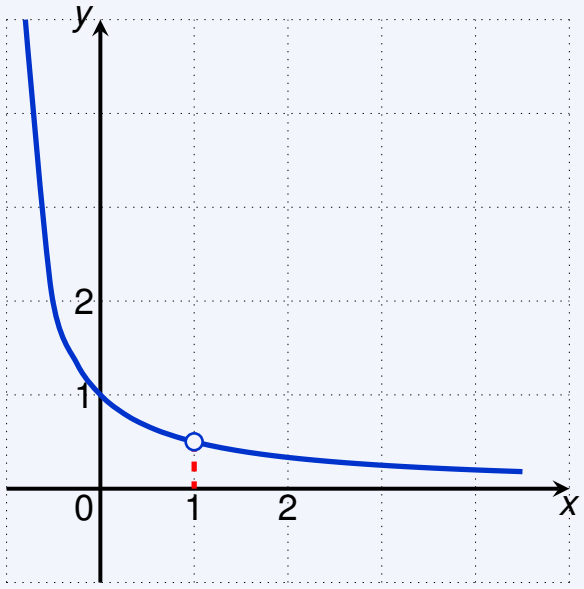

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From these values we guess that $\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$.

Limit: Examples

Guess the value of $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$$

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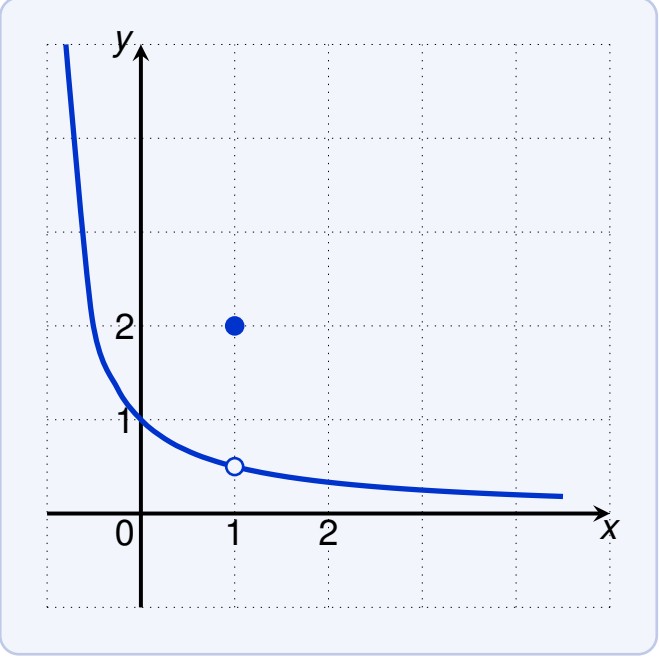

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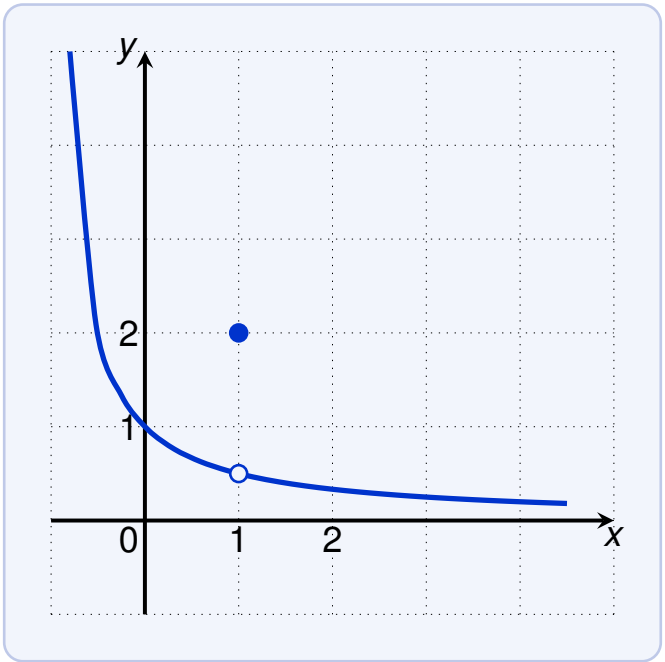

As on the previous slide $\lim_{x \to 1} g(x) = 0.5$.
(recall that $g(1)$ does not matter for $\lim_{x \to 1} g(x)$).

Limit: Examples

Guess the value of

$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

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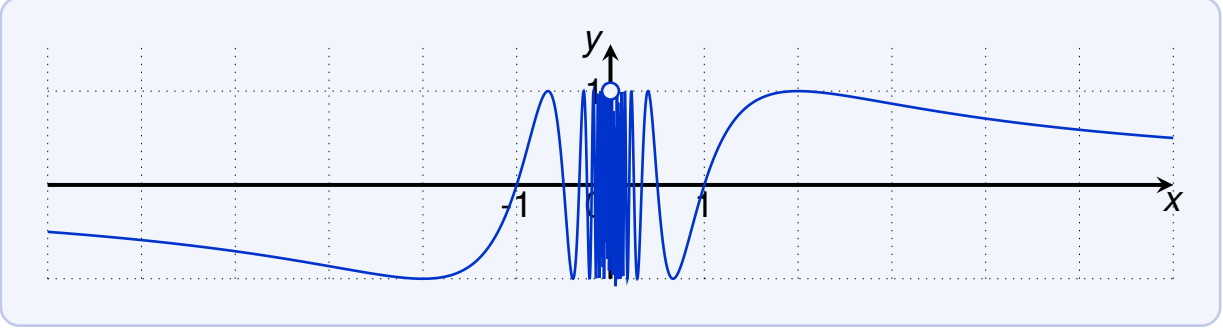

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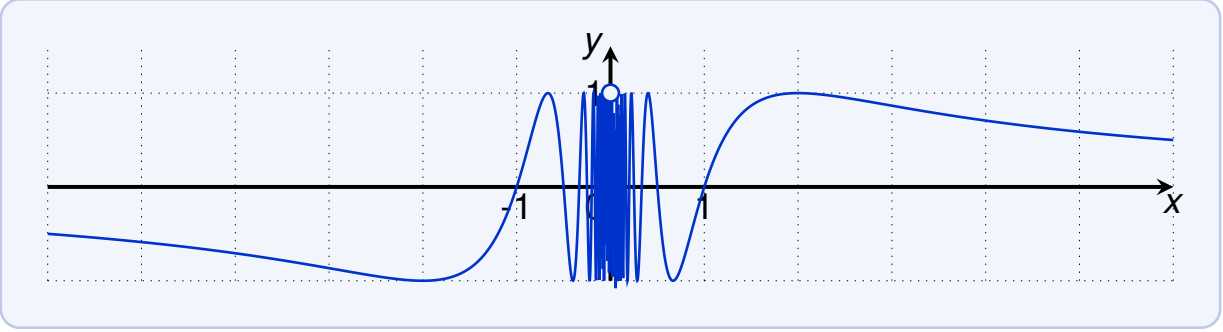

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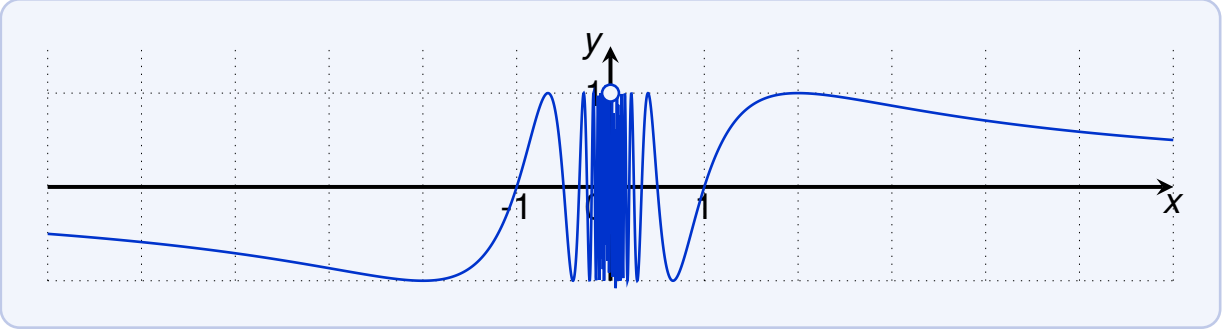

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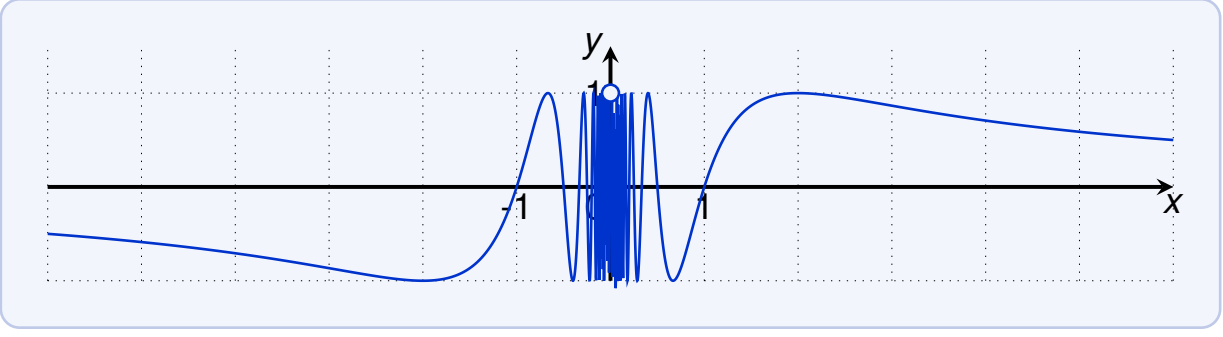

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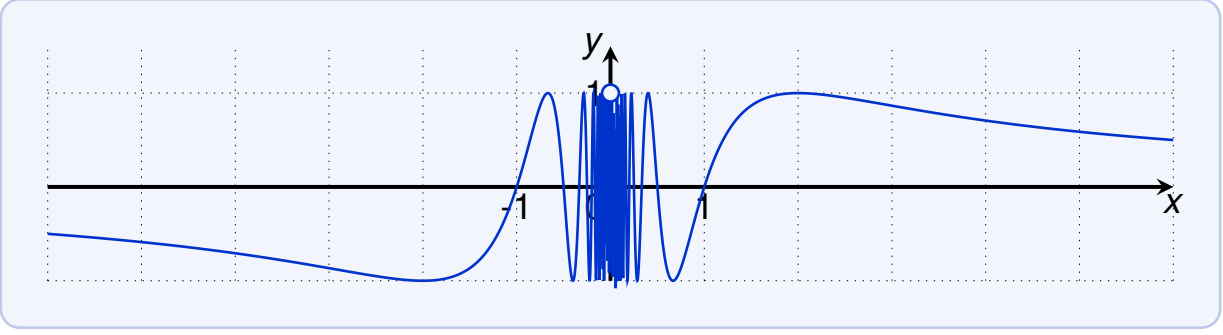

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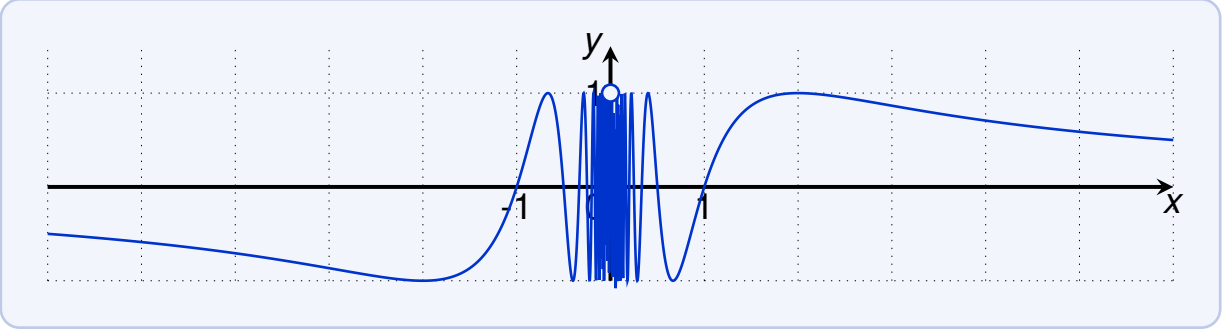

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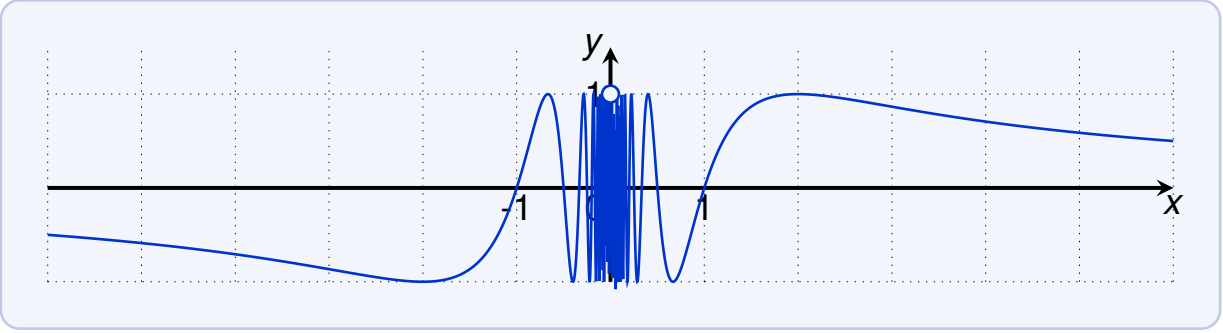

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$\sin(\frac{\pi}{x}) = 1$ for arbitrarily small x; e.g. $x = \frac{1}{2.5}, \frac{1}{4.5}, \frac{1}{6.5}, \ldots$
Hence: The limit $\lim_{x \to 0} \sin \frac{\pi}{x}$ does not exist.

Limit: Caution with Calculators

Guess the value of

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

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x	$f(x)$
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± 0.5	0.16553
± 0.1	0.16662
± 0.01	0.16667
± 0.0001	0.20000
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Is the limit 0?

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Problem: calculator gives wrong values!
For small x it rounds $\sqrt{x^2 + 9} - 3$ to 0.

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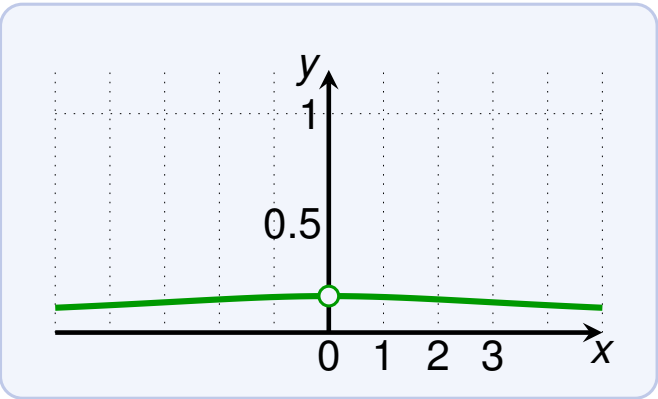

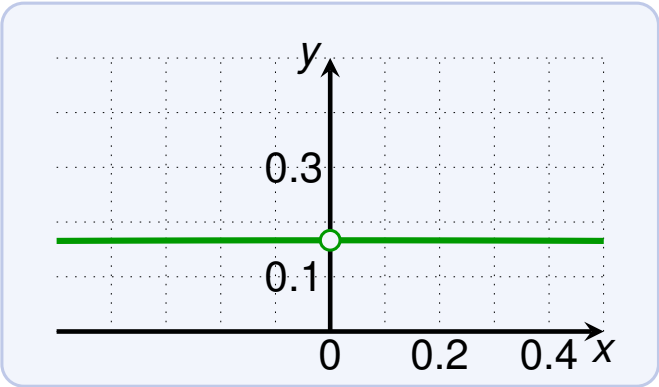

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Is the limit 0? NO

Problem: calculator gives wrong values!
For small x it rounds $\sqrt{x^2+9}-3$ to 0.

The correct limit is $\frac{1}{6} = 0.166666\ldots$

Limit: Examples

Guess the value of

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10000} \right)$$

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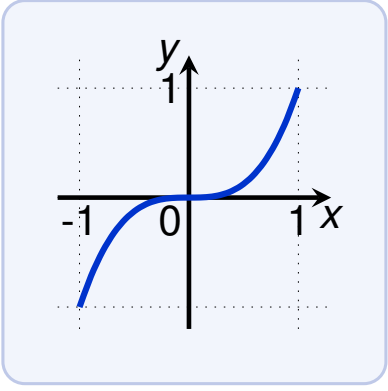

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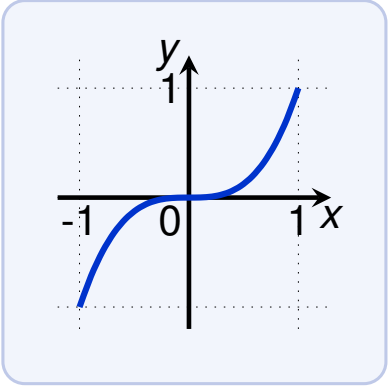

x	$f(x)$
1	1.000028
0.5	0.124920
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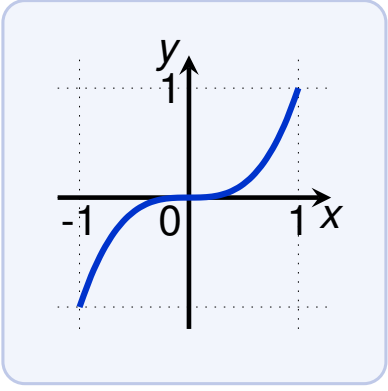

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Looks like the limit is 0.

Limit: Examples

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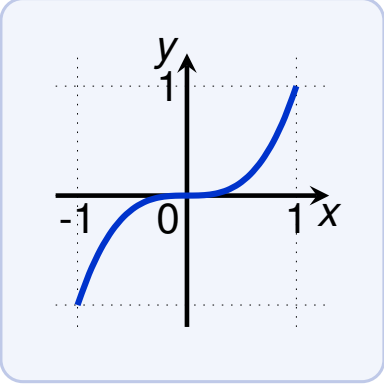

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Looks like the limit is 0. But if we continue:

x	$f(x)$
0.005	0.00010009
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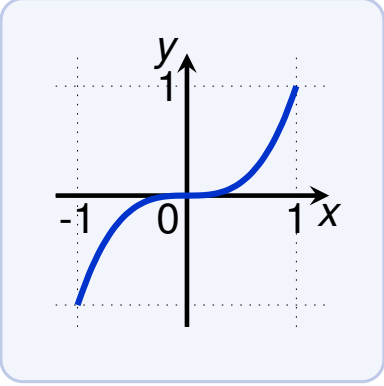

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We see actually that:

The value of the limit is 0.0001.

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Determining limits via calculators is a bad idea!

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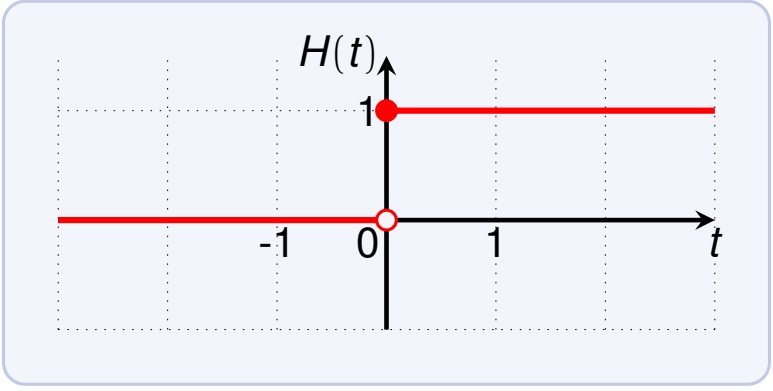
We need to compute limits precisely using limits laws...

Limit: Examples

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

What is $\lim_{t \to 0} H(t)$?



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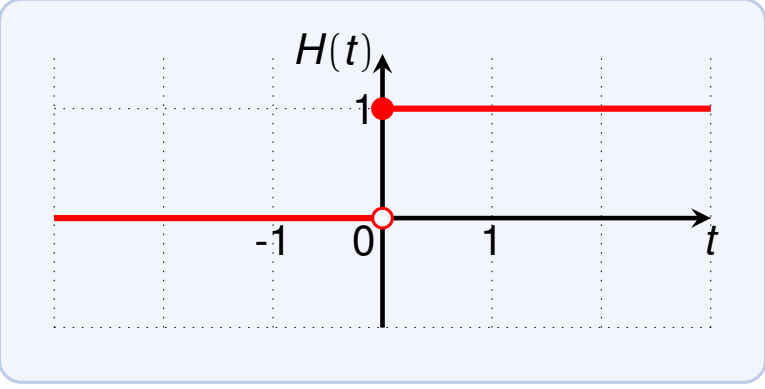

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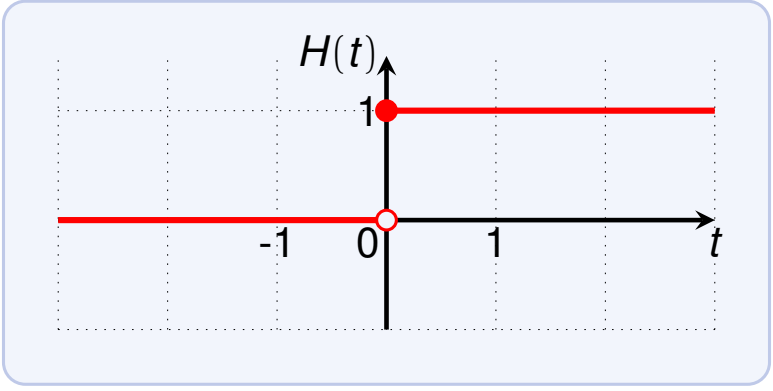

- As t approaches 0 from the left, $H(t)$ approaches 0.
- As t approaches 0 from the right, $H(t)$ approaches 1.

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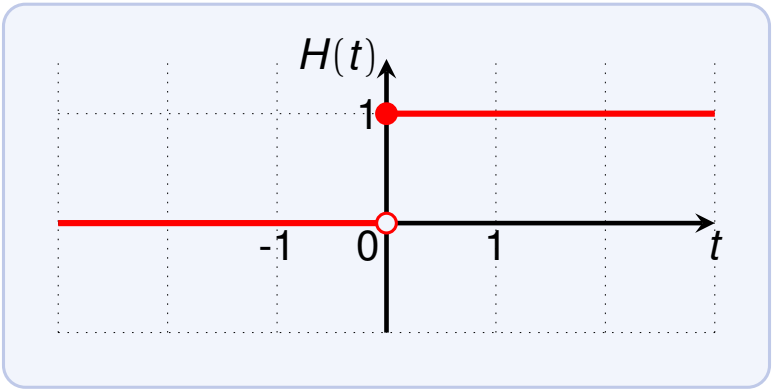
What is $\lim_{t \to 0} H(t)$?



- As t approaches 0 from the left, $H(t)$ approaches 0.
- As t approaches 0 from the right, $H(t)$ approaches 1.

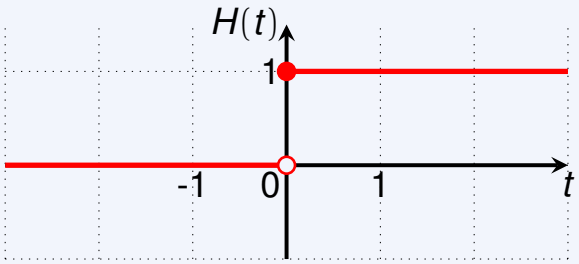
Thus there is not single number that $H(t)$ approaches.

The limit $\lim_{t \to 0} H(t)$ does not exist.

One-Sided Limits (From the Left)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



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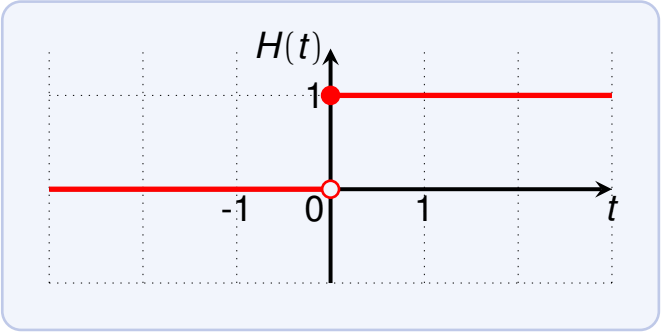

$H(t)$ approaches 0 as t approaches 0 from the left. We write:

$$\lim_{t \to 0^-} H(t) = 0$$

The symbol $t \to 0^-$ indicates that we consider only values $t < 0$.

One-Sided Limits (From the Left)

The function H is defined by

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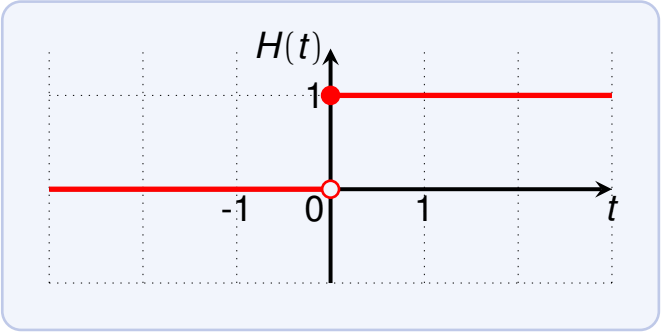


$H(t)$ approaches 0 as t approaches 0 from the left. We write:

$$\lim_{t \to 0^-} H(t) = 0$$

The symbol $t \to 0^-$ indicates that we consider only values $t < 0$.

We write $\lim_{x \to a^-} f(x) = L$ and say

"the left-hand limit of $f(x)$, as x approaches a, is L", *or*

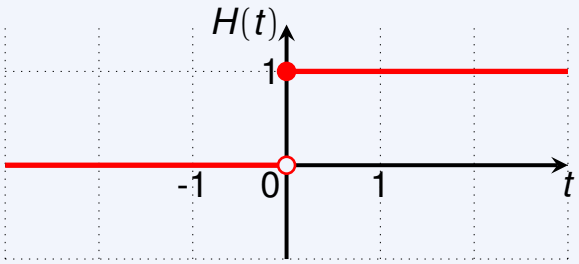
"the limit of $f(x)$, as x approaches a from the left, is L"

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

One-Sided Limits (From the Right)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



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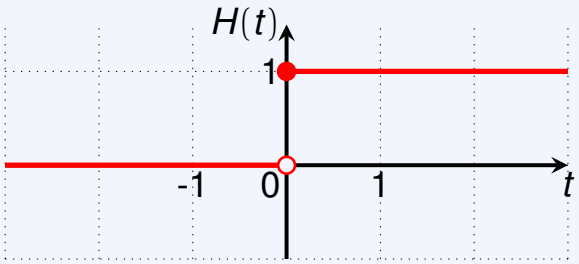


$H(t)$ approaches 1 as t approaches 0 from the right. We write:

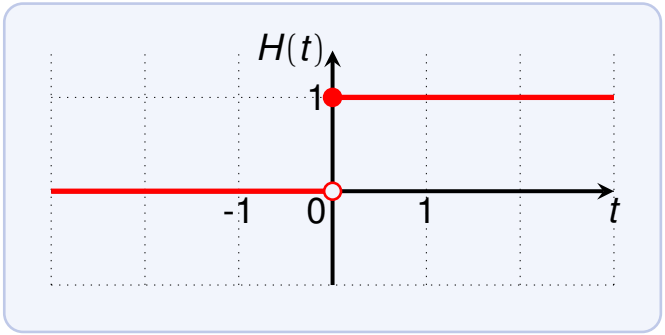
$$\lim_{t \to 0^+} H(t) = 1$$

The symbol $t \to 0^+$ indicates that we consider only values $t > 0$.

One-Sided Limits (From the Right)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$H(t)$ approaches 1 as t approaches 0 from the right. We write:

$$\lim_{t \to 0^+} H(t) = 1$$

The symbol $t \to 0^+$ indicates that we consider only values $t > 0$.

We write $\lim_{x \to a^+} f(x) = L$ and say

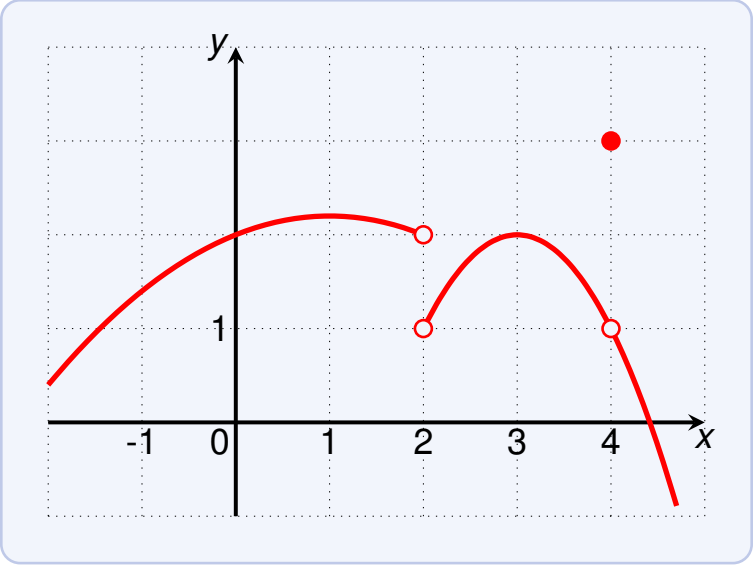
"the right-hand limit of $f(x)$, as x approaches a, is L", *or*

"the limit of $f(x)$, as x approaches a from the right, is L"

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

One-Sided Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

One-Sided Limits: Example

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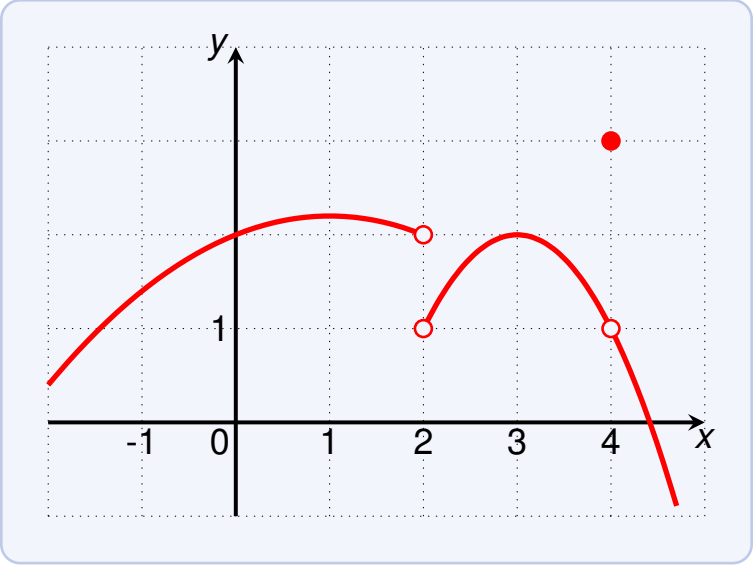
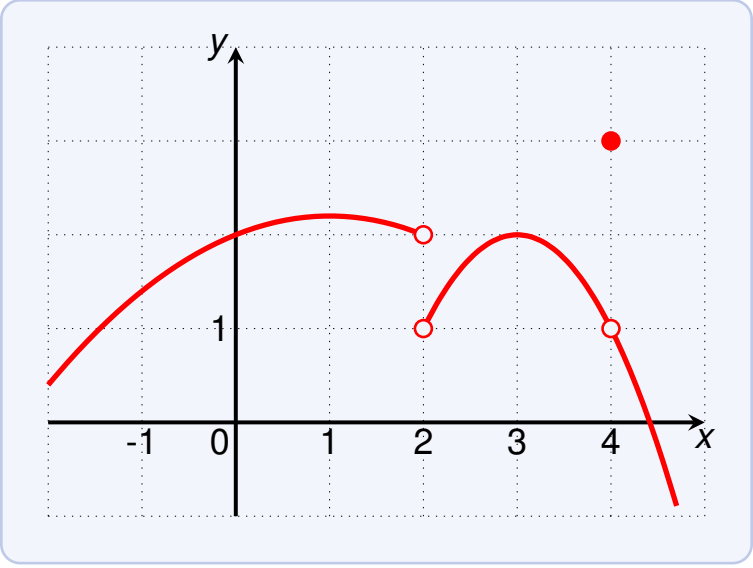


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

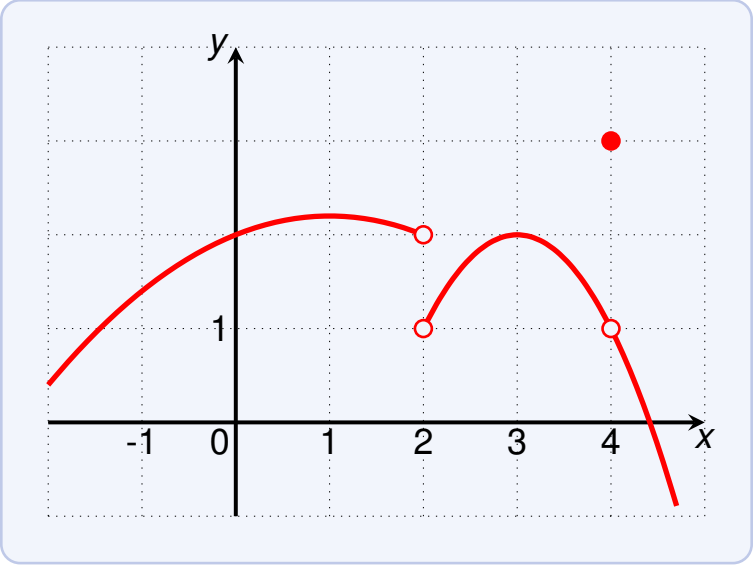


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$

One-Sided Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
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One-Sided Limits: Example

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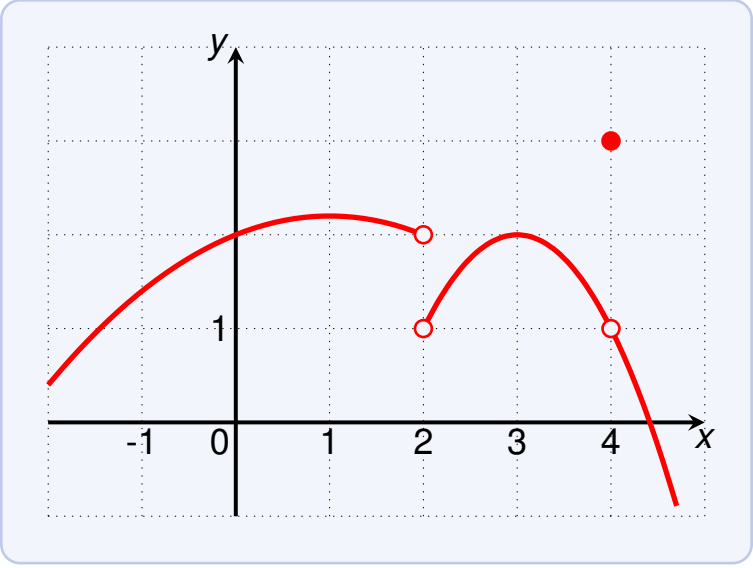

Use the graph to estimate the following values:

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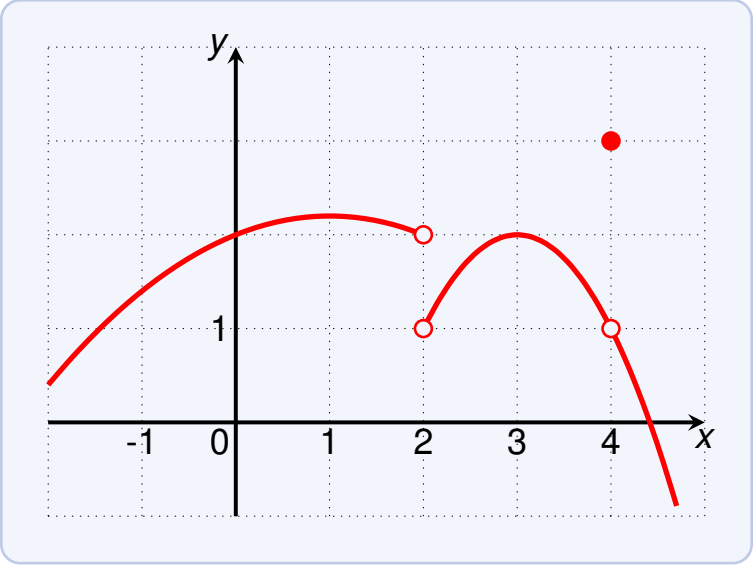

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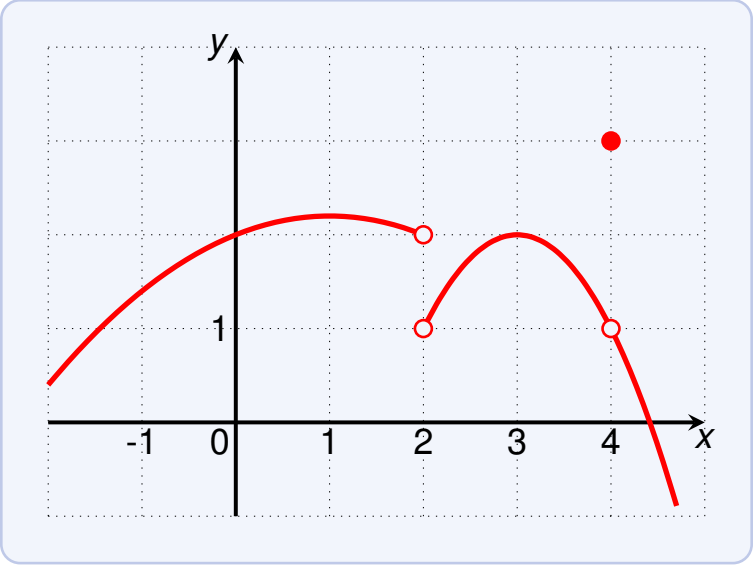

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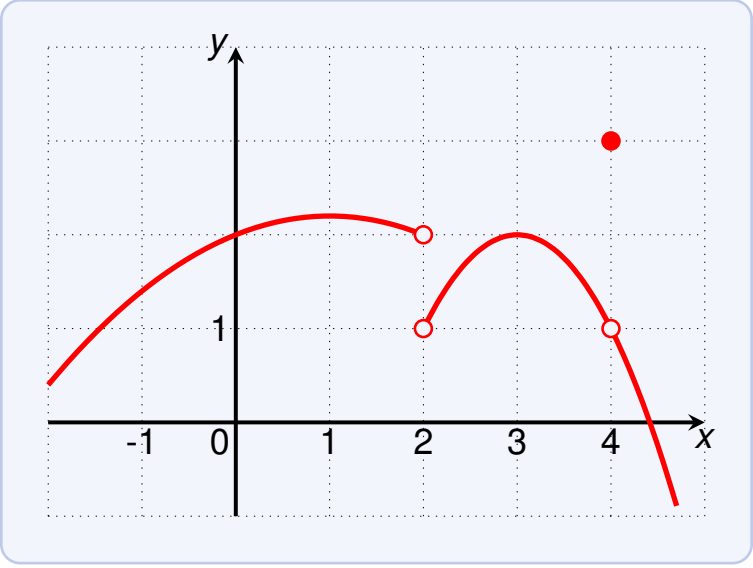

Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = 1$
- $\lim_{x \to 2}$ does not exist
- $\lim_{x \to 4^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

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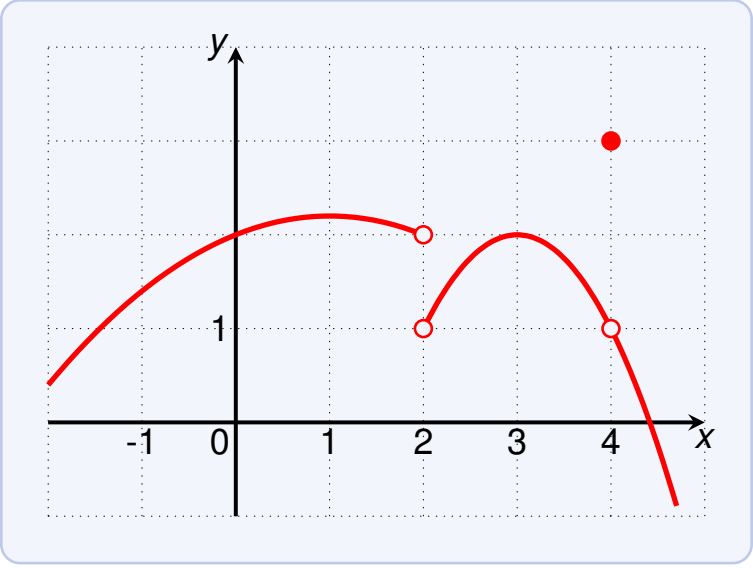

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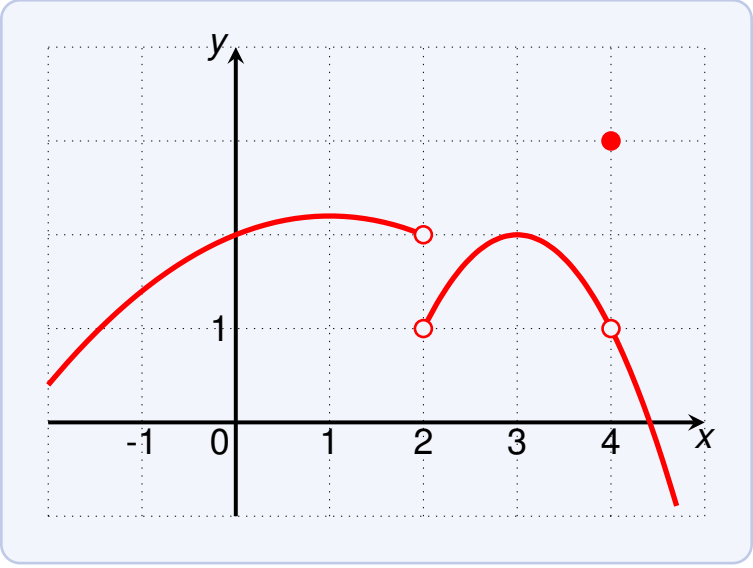

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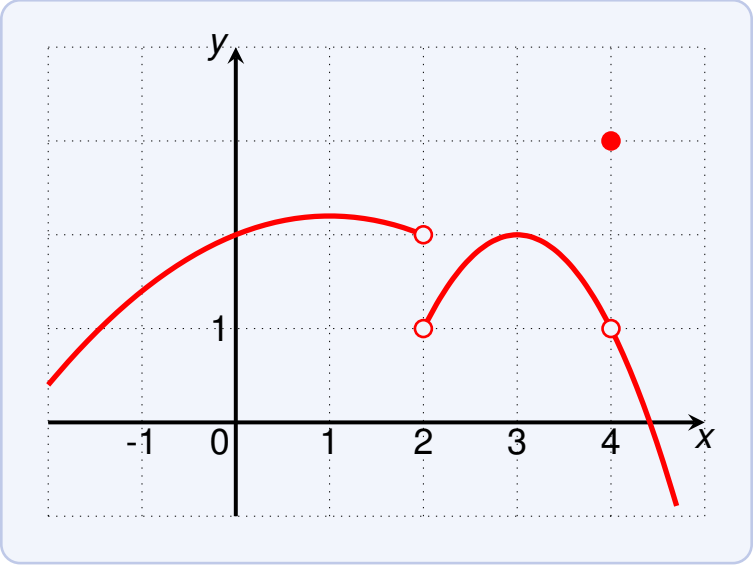

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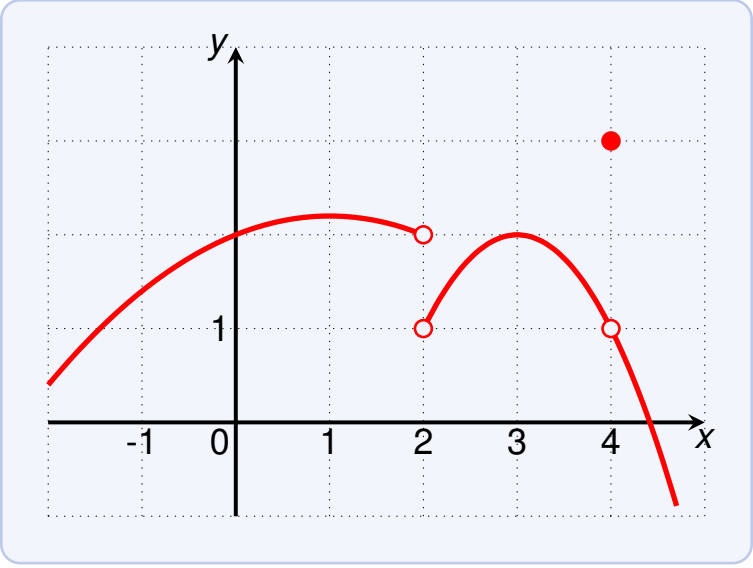

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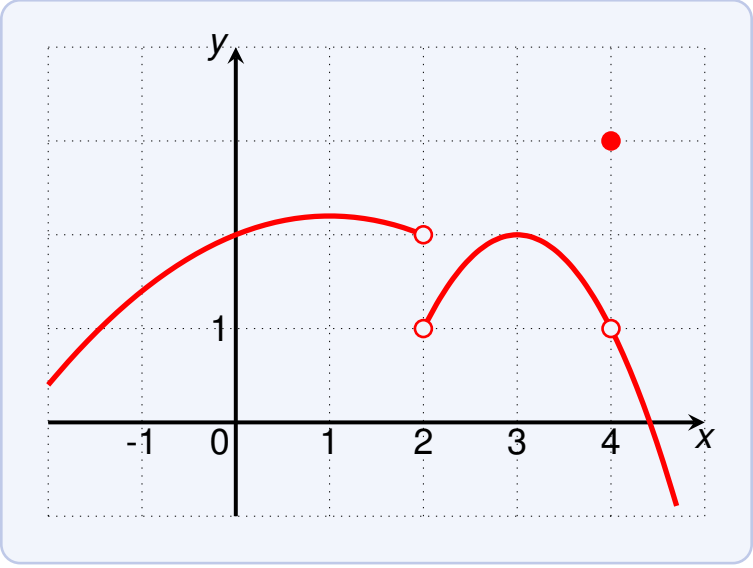

Use the graph to estimate the following values:

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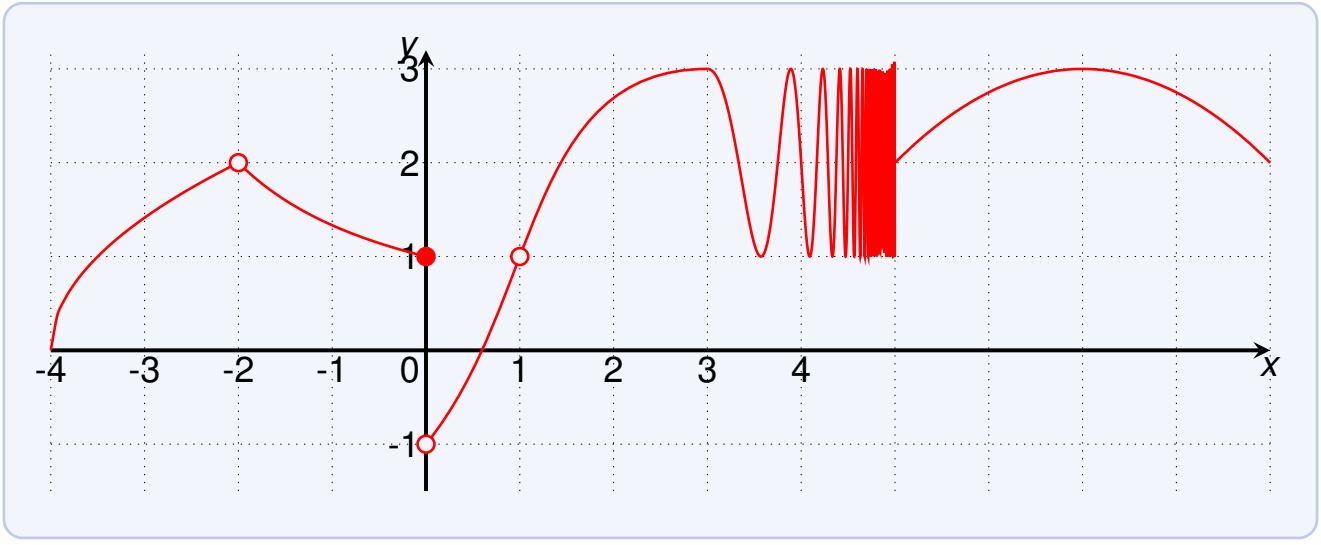
Infinite Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

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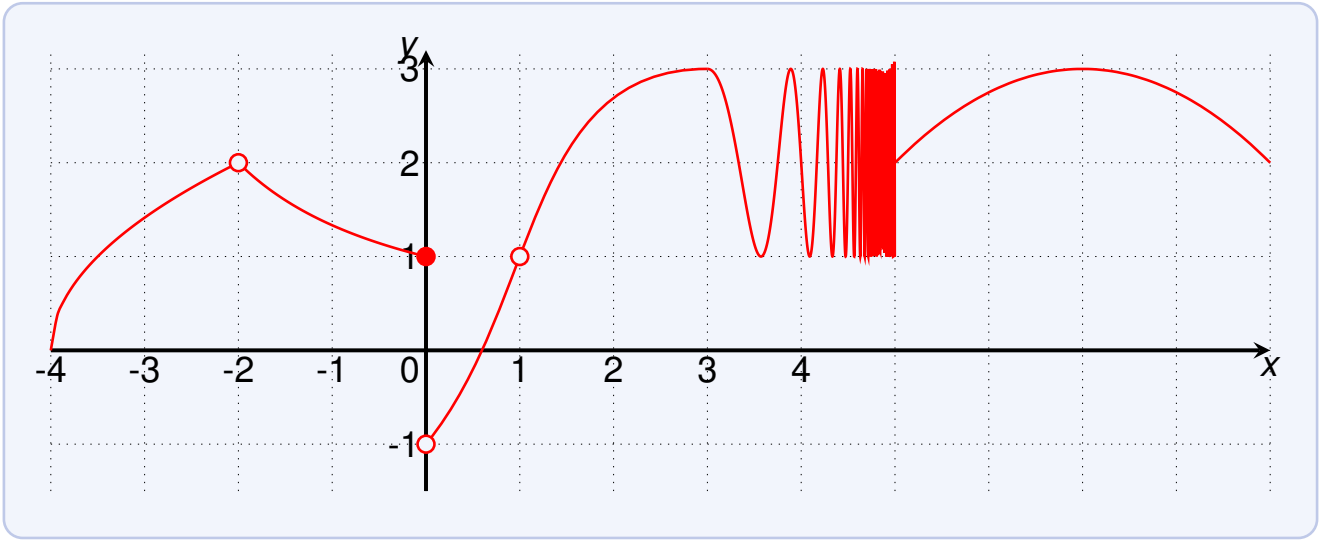

Use the graph to estimate the following values:

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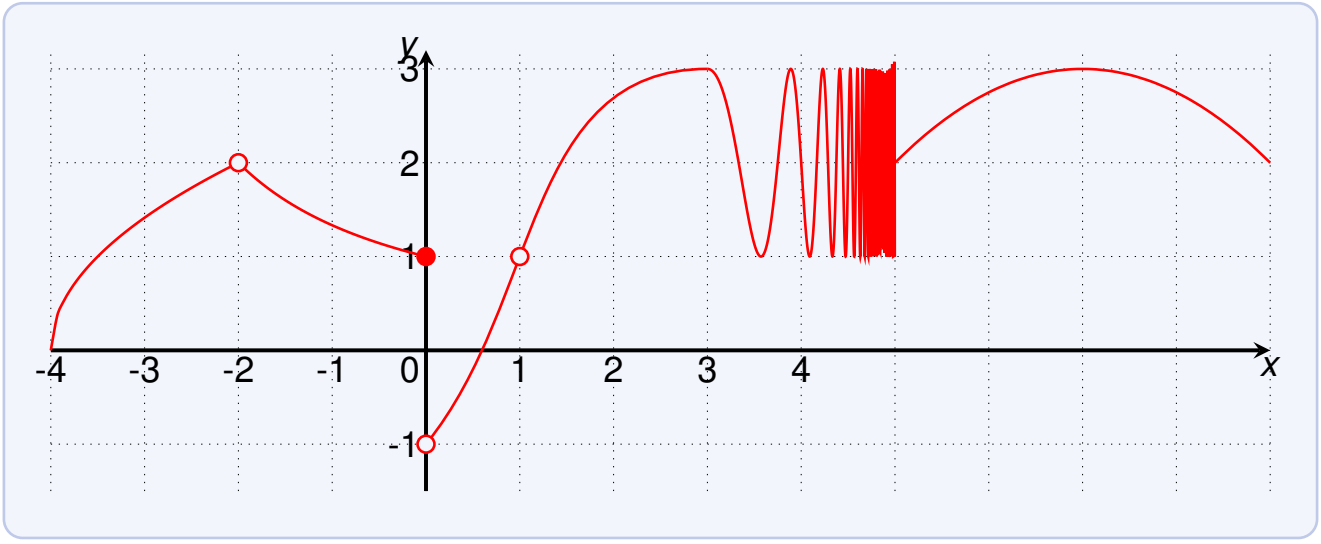

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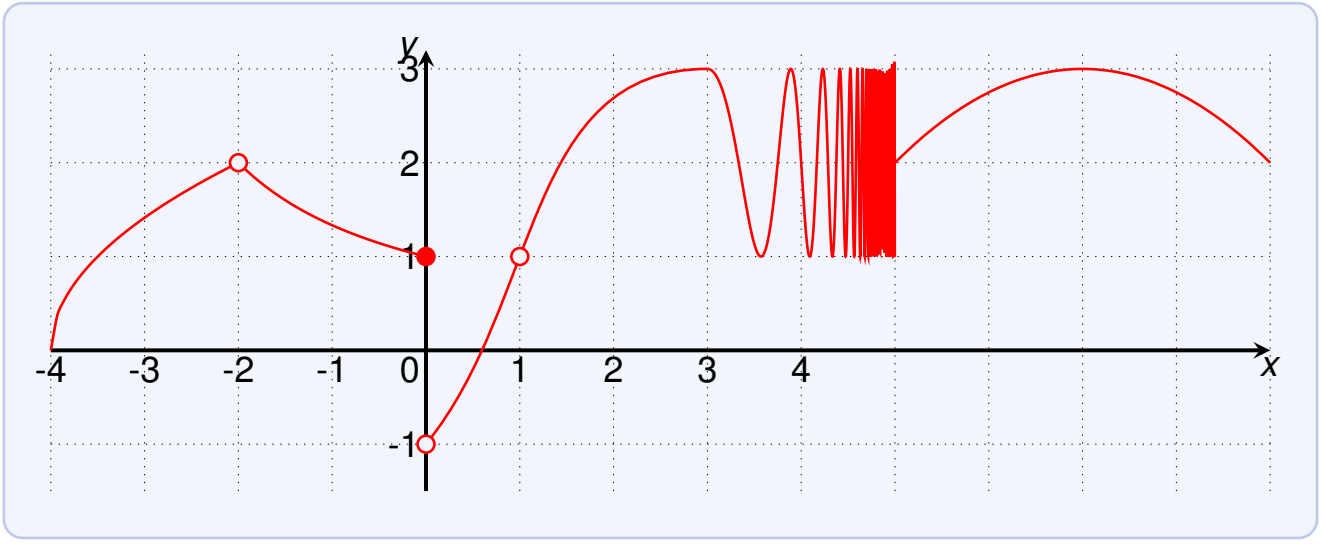

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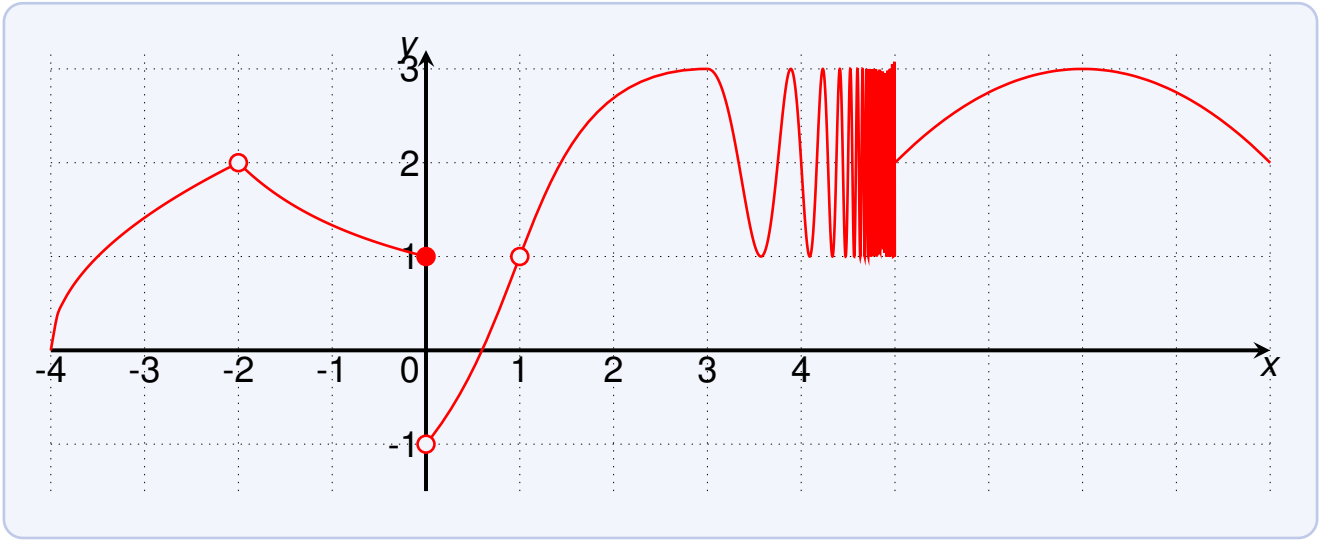

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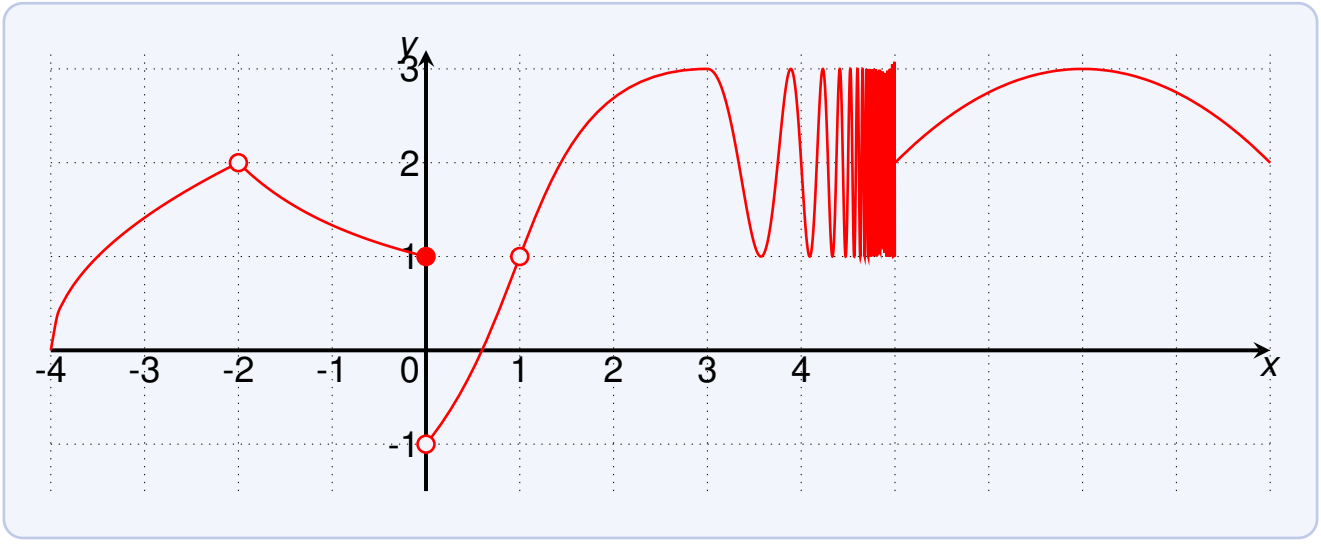

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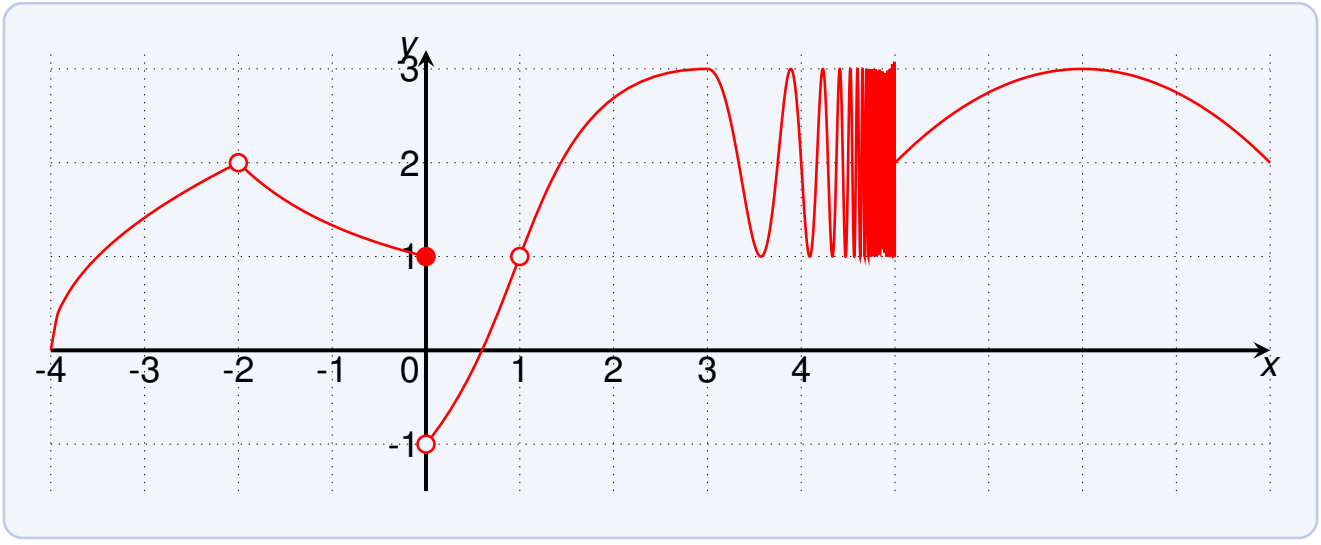

Use the graph to estimate the following values:

- $\lim_{x \to 3^-} g(x) = 3$
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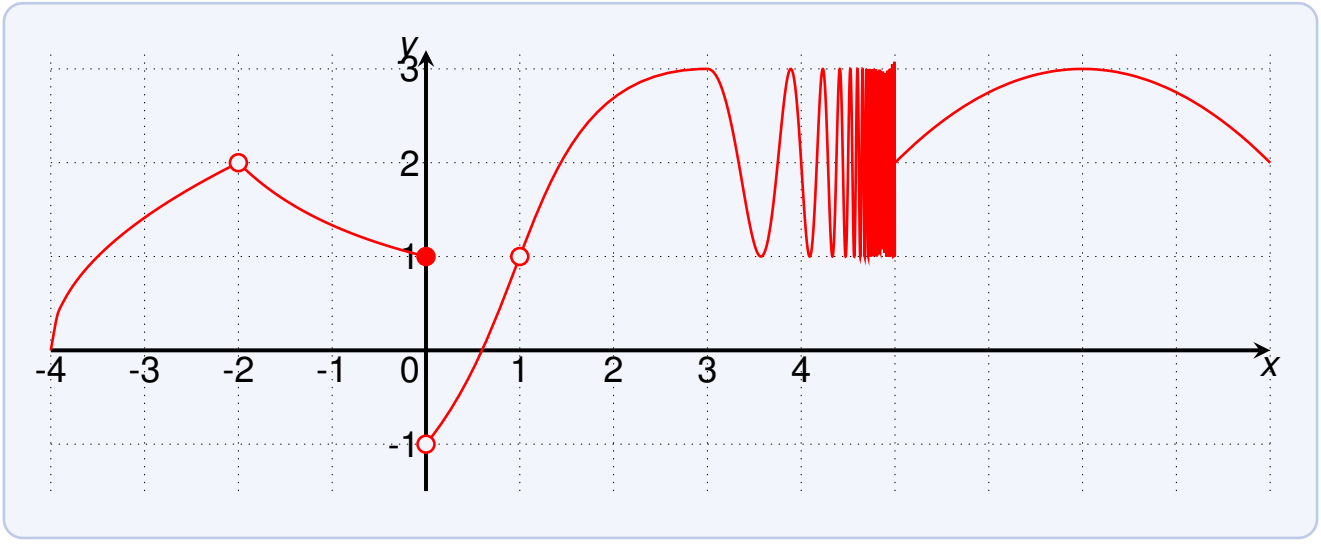
Infinite Limits: Example

Consider the following graph of function $g(x)$:



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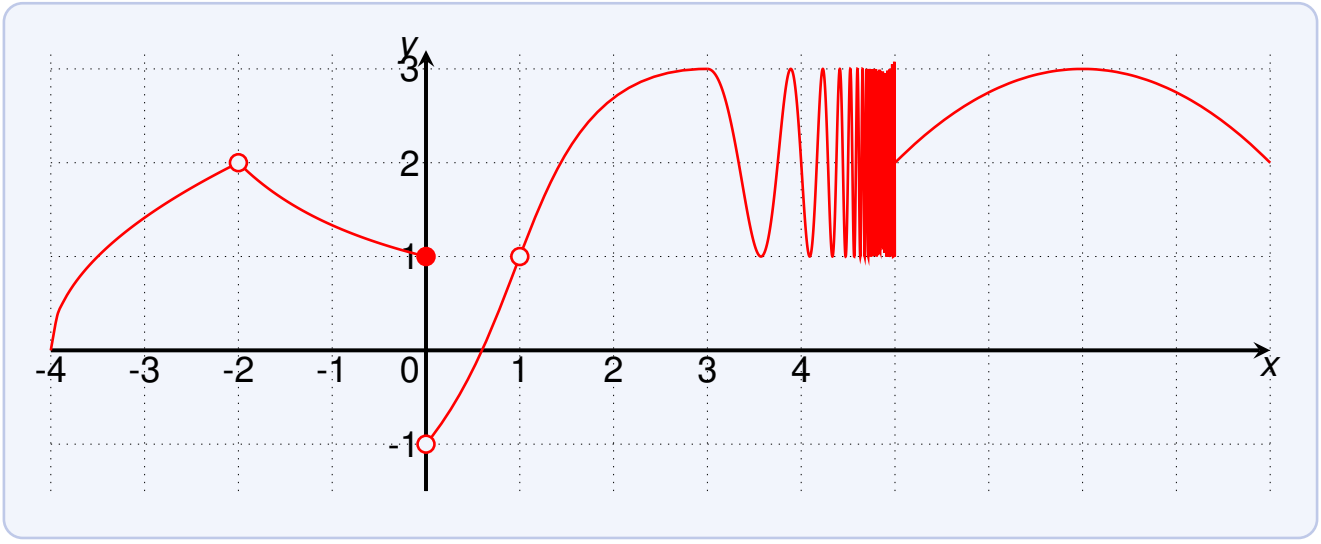

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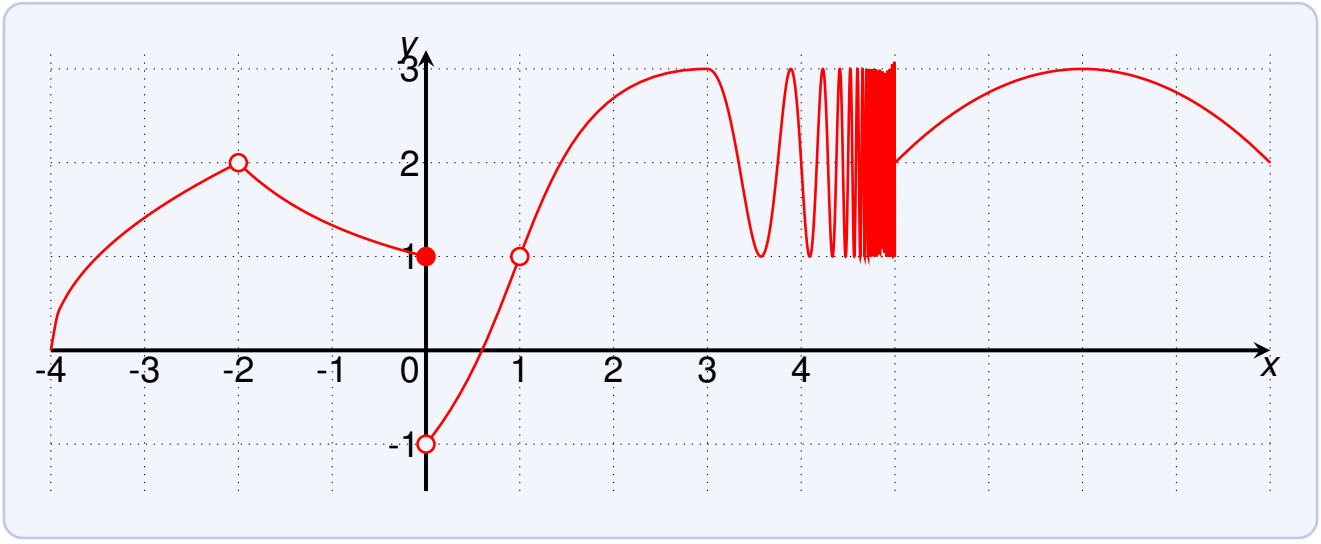
- $g(1) =$

Infinite Limits: Example

Consider the following graph of function $g(x)$:



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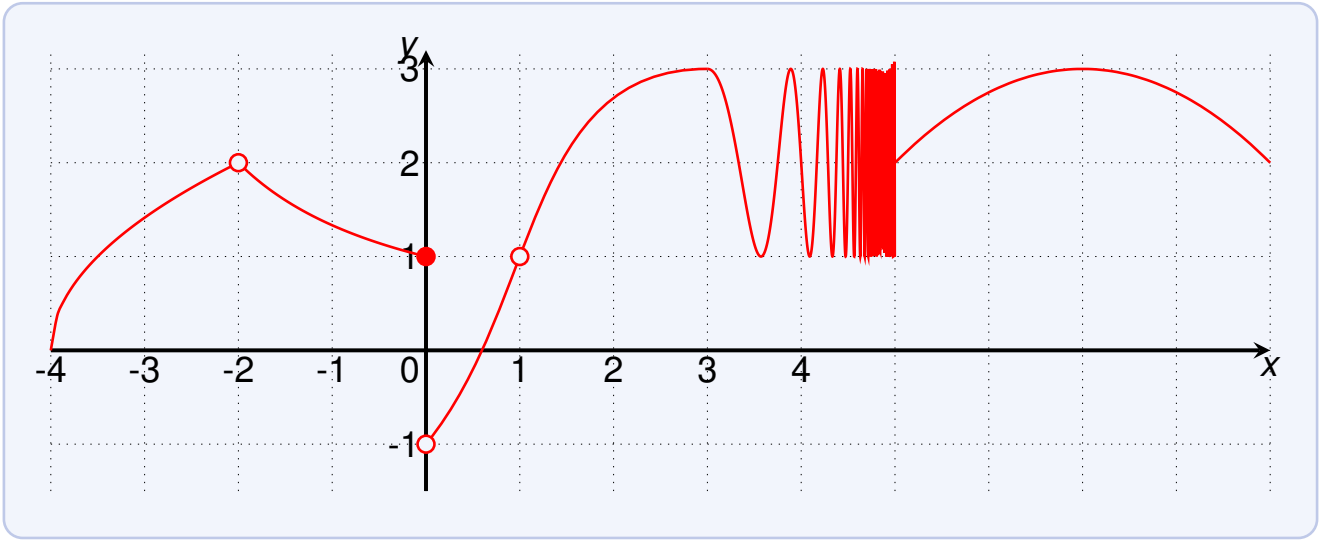

Use the graph to estimate the following values:

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Infinite Limits: Example

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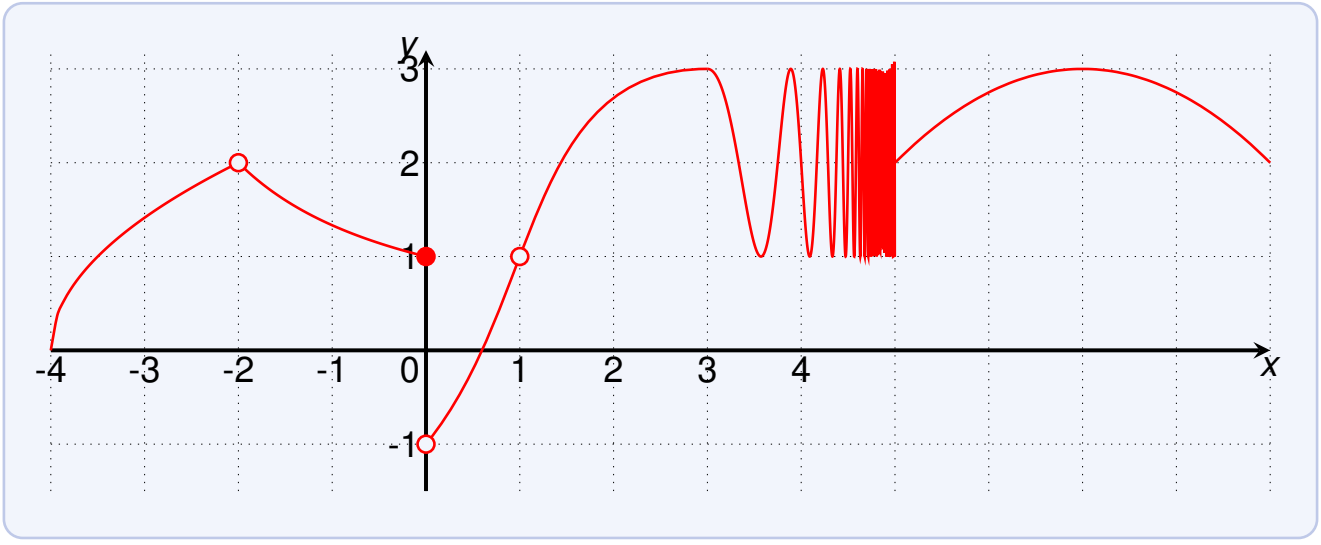

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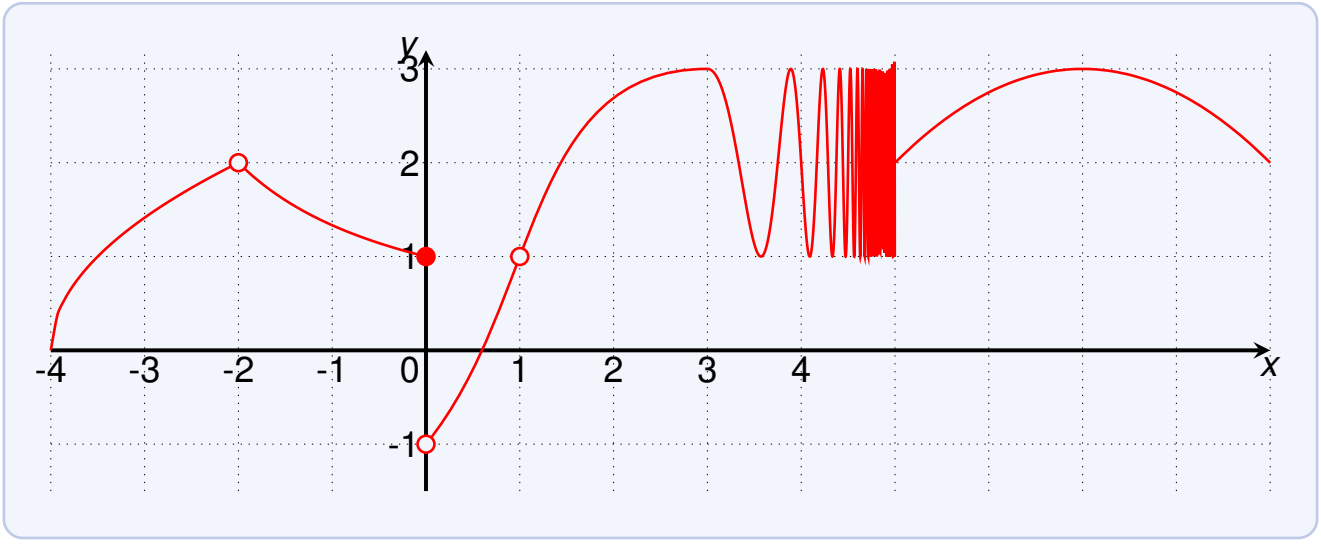

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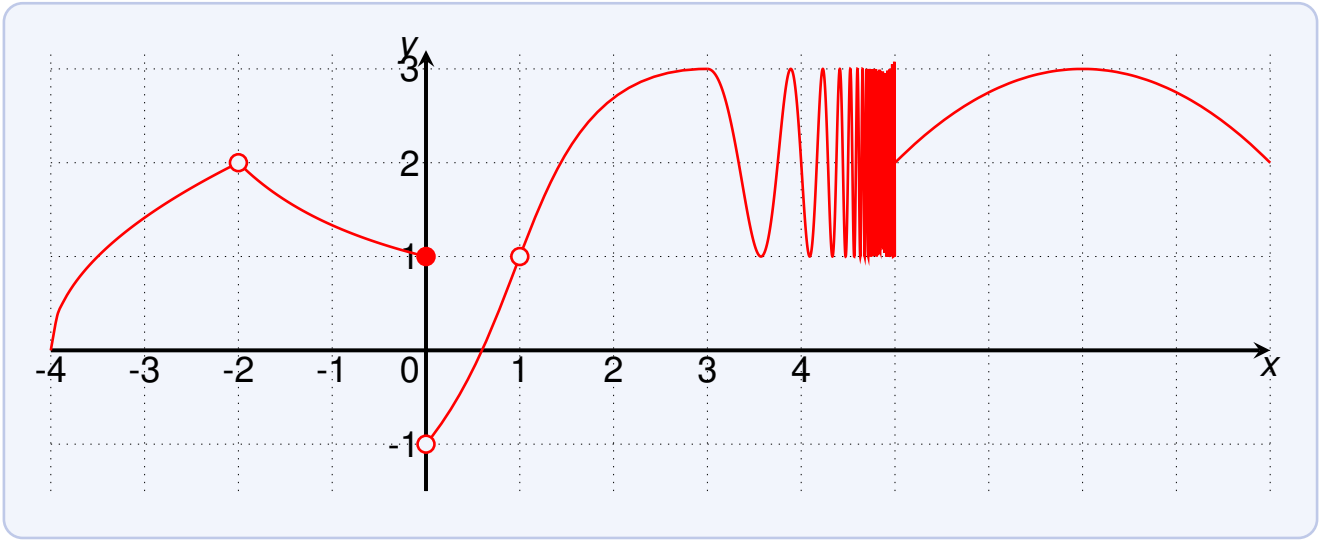

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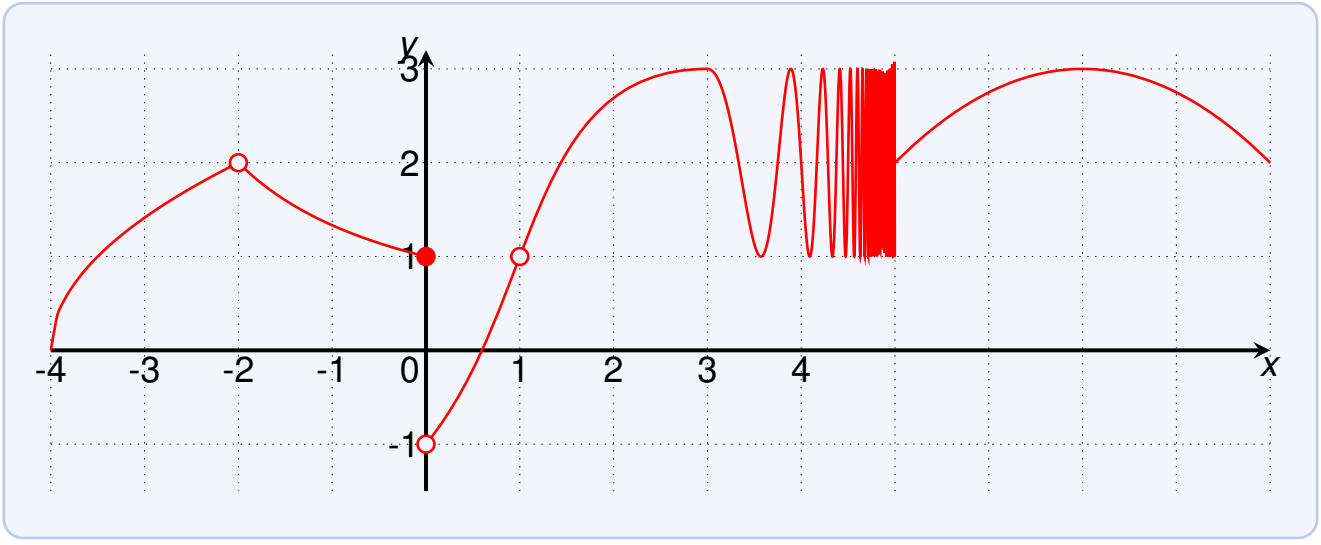
Infinite Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

- $\lim_{x \to 3^-} g(x) = 3$
- $\lim_{x \to 3^+} g(x) = 3$
- $\lim_{x \to 3} g(x) = 3$
- $\lim_{x \to 1} g(x) = 1$
- $g(1) =$ undefined
- $g(0) = 1$
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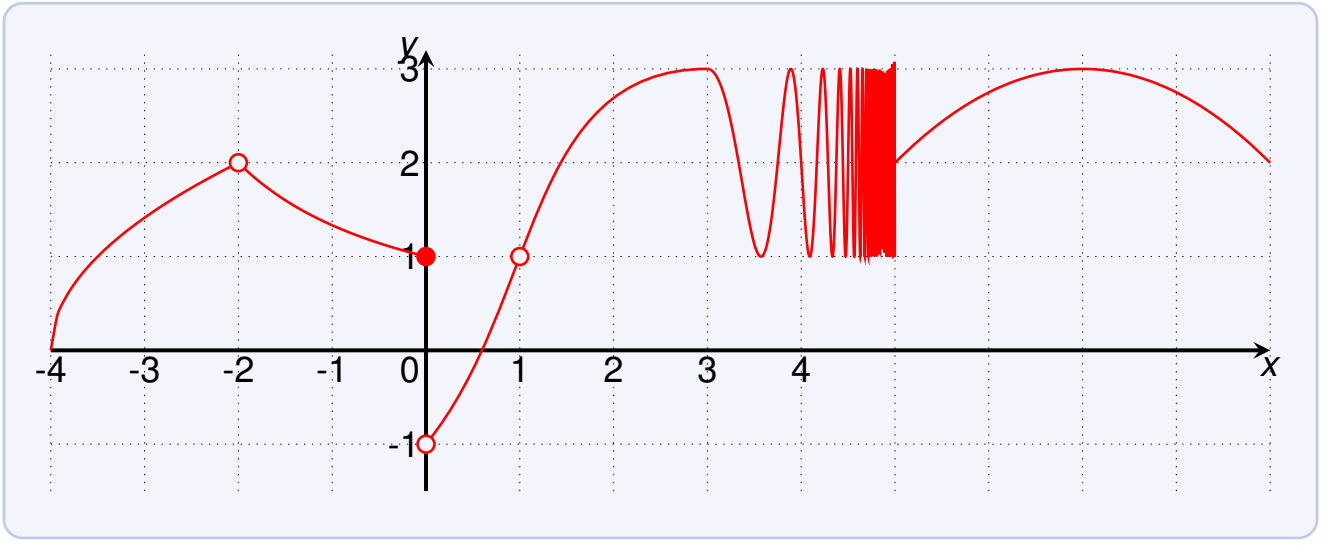

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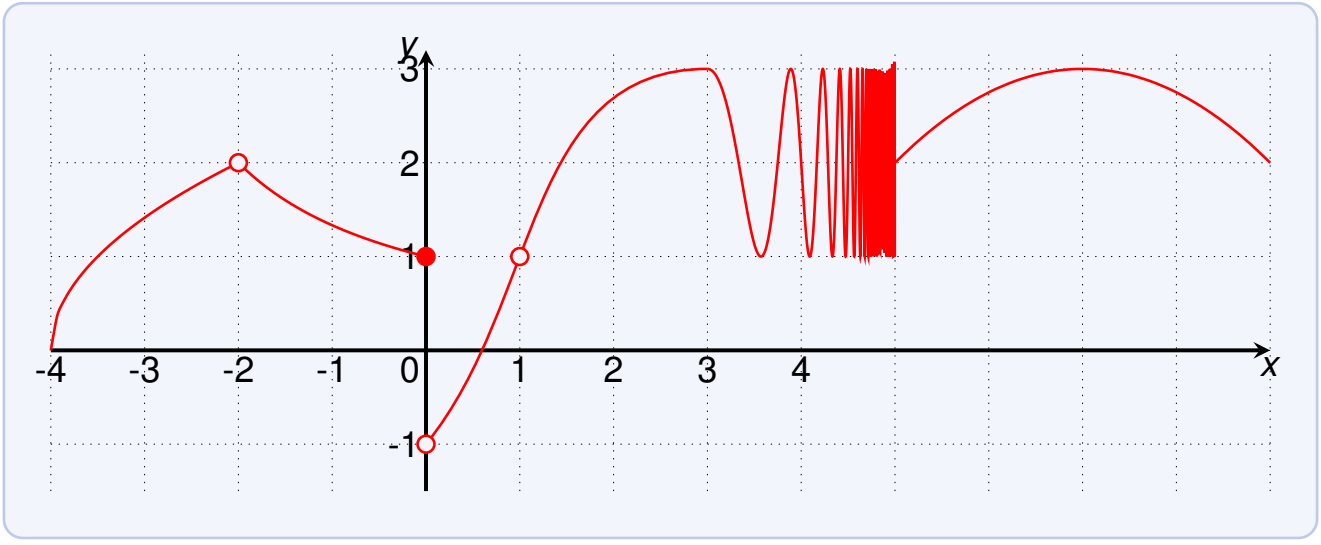

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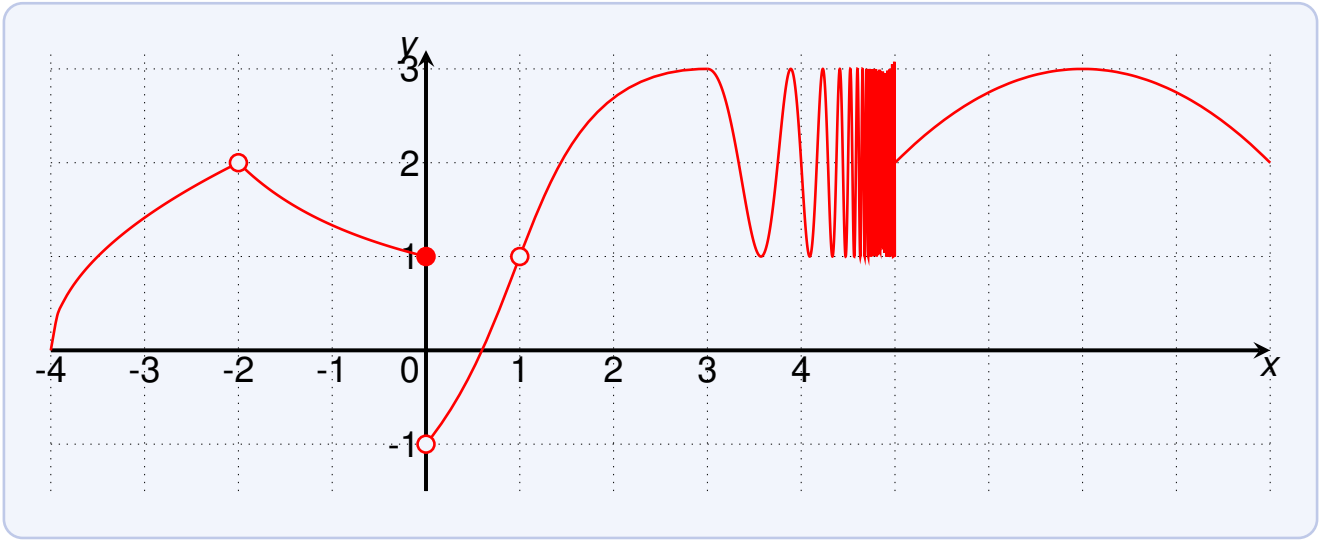

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- $\lim_{x \to 0} g(x) =$ does not exist

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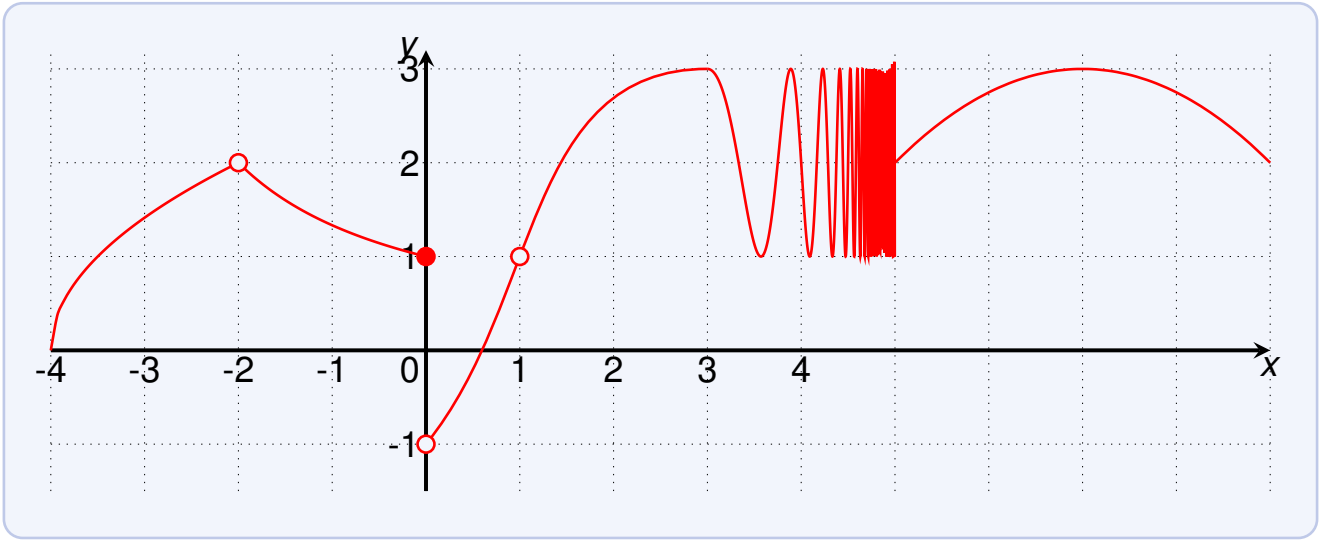

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- $\lim_{x \to 0} g(x) =$ does not exist
- $\lim_{x \to 5^-} g(x) =$

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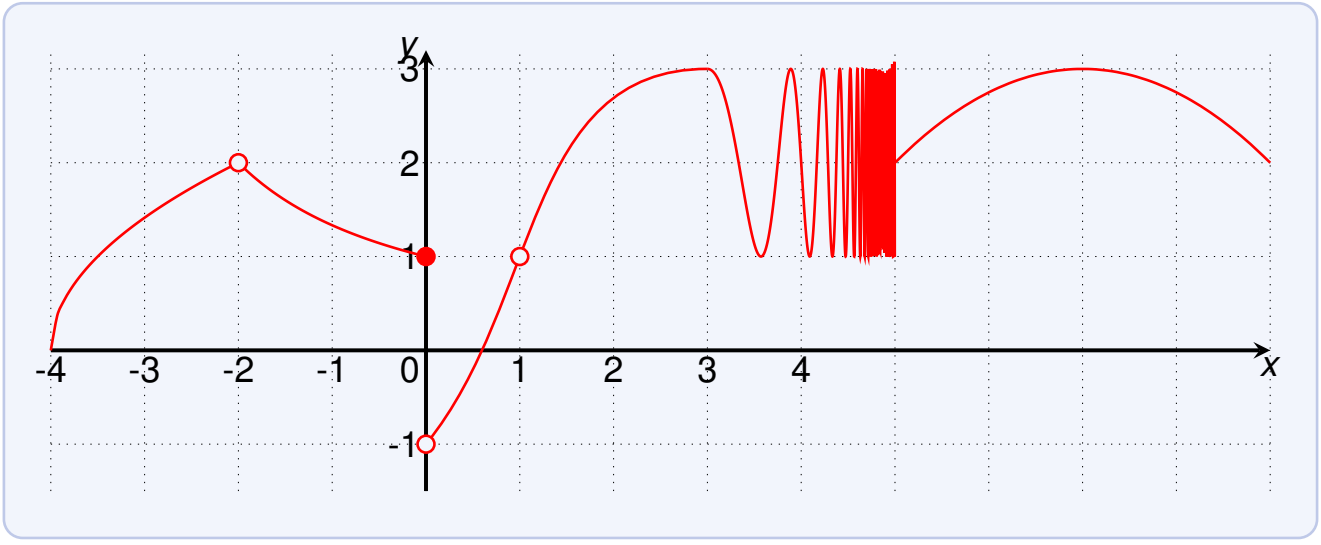

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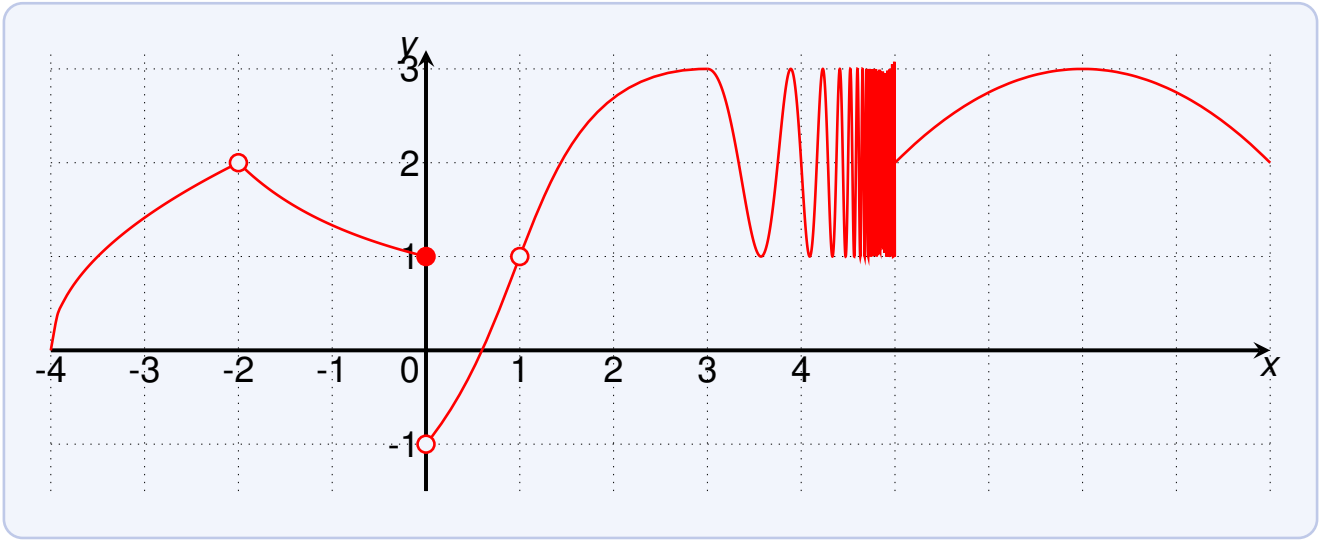

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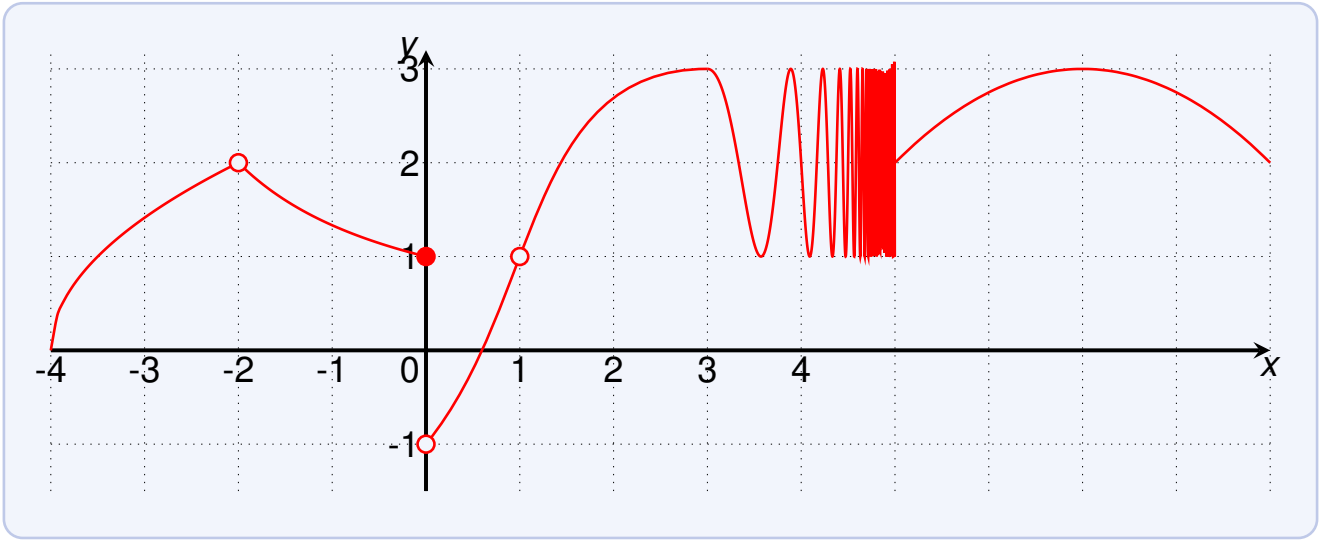

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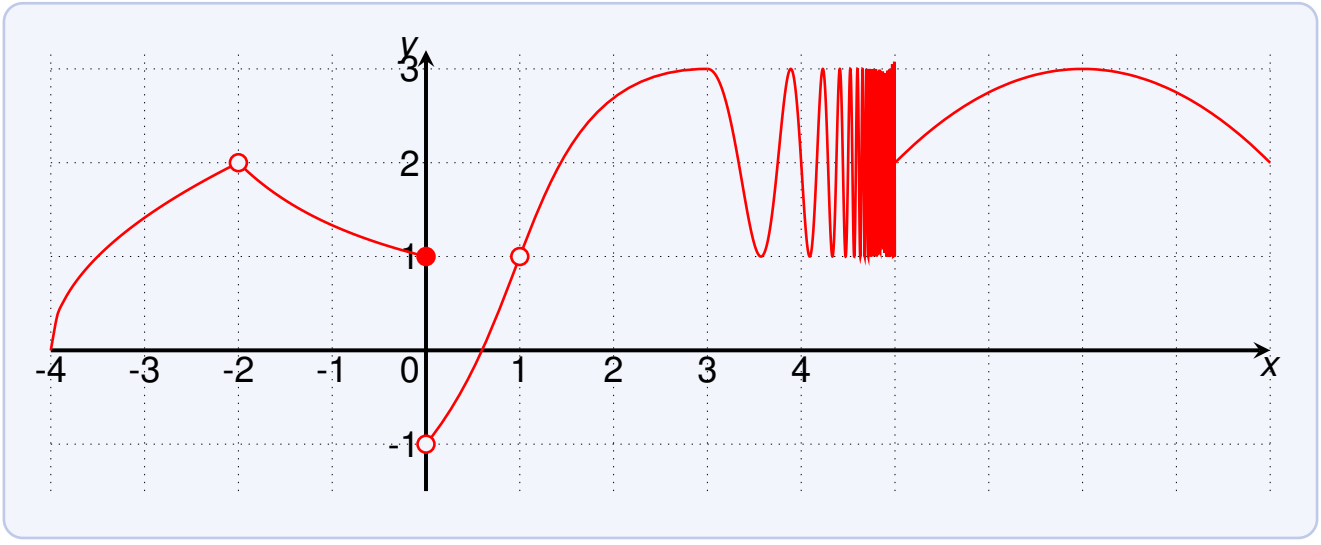
- $g(1) =$ undefined
- $g(0) = 1$
- $\lim_{x \to 0^-} g(x) = 1$
- $\lim_{x \to 0^+} g(x) = -1$
- $\lim_{x \to 0} g(x) =$ does not exist
- $\lim_{x \to 5^-} g(x) =$ does not exist
- $\lim_{x \to 5^+} g(x) = 2$

Infinite Limits: Example

Consider the following graph of function $g(x)$:



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Infinite Limits: Example

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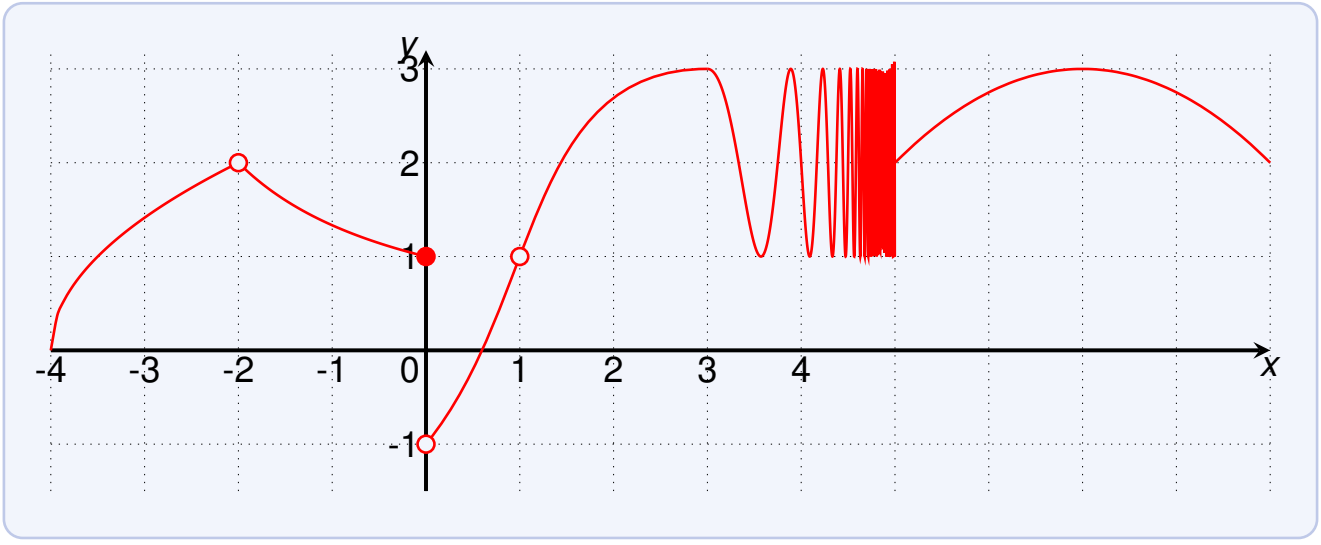

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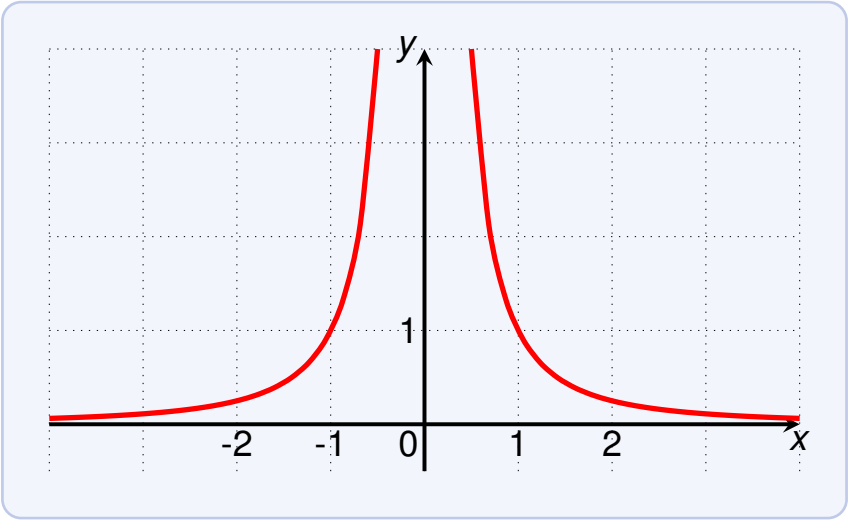
Infinite Limits

We consider the function $\frac{1}{x^2}$. What is $\lim_{x \to 0} \frac{1}{x^2}$?



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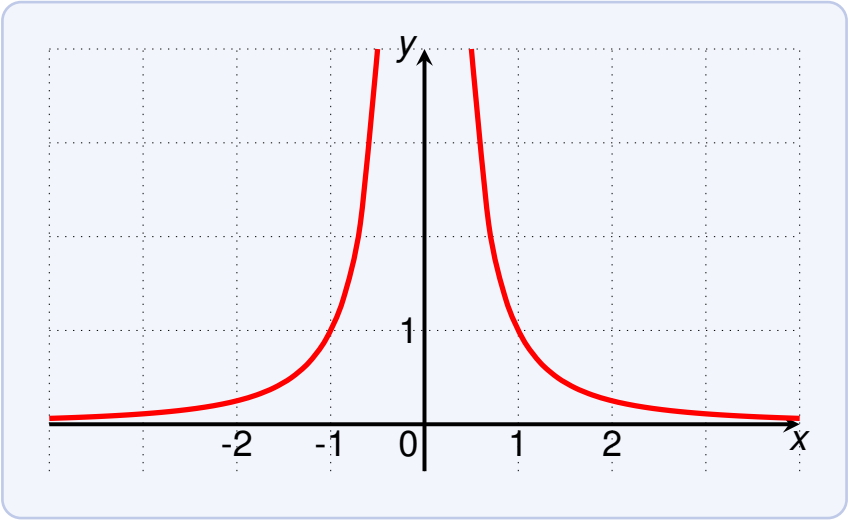

As x becomes close to 0, $\frac{1}{x^2}$ becomes very large.

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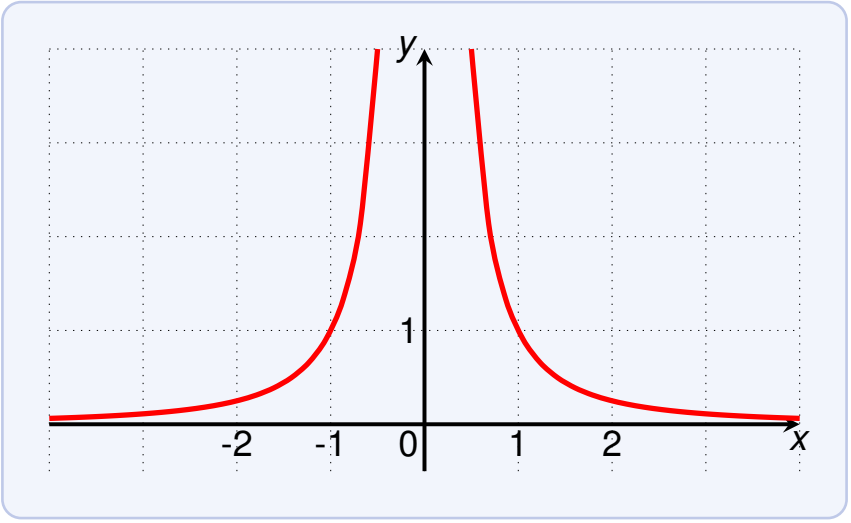

As x becomes close to 0, $\frac{1}{x^2}$ becomes very large. The values do not approach a number, so $\lim_{x \to 0} \frac{1}{x^2}$ does not exist!

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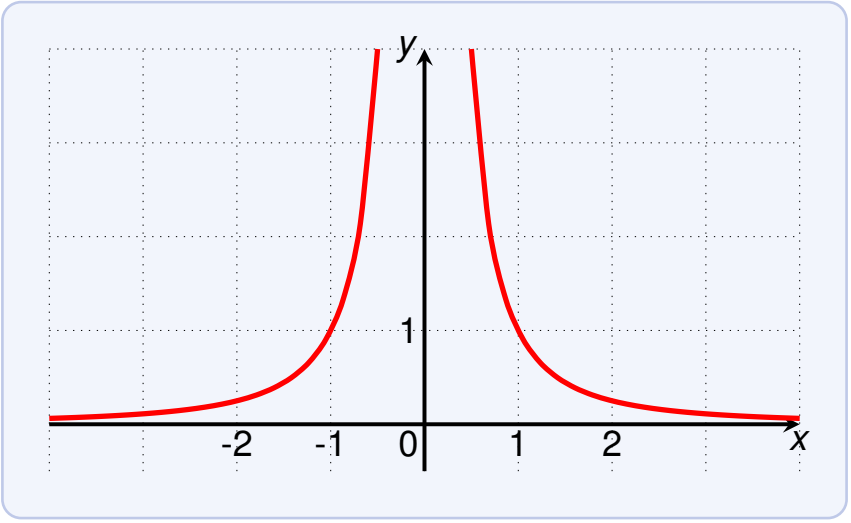


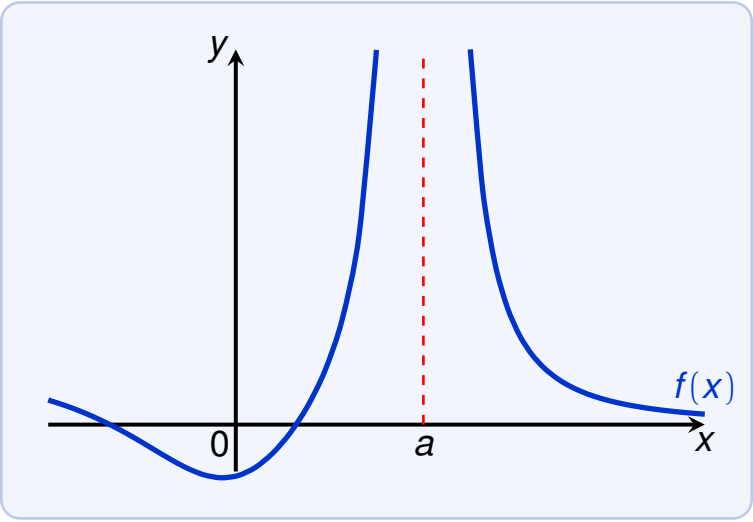
As x becomes close to 0, $\frac{1}{x^2}$ becomes very large. The values do not approach a number, so $\lim_{x \to 0} \frac{1}{x^2}$ does not exist!

Nevertheless, in this case, we write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

to indicate that the values become larger and larger.

Infinite Limits: Definition



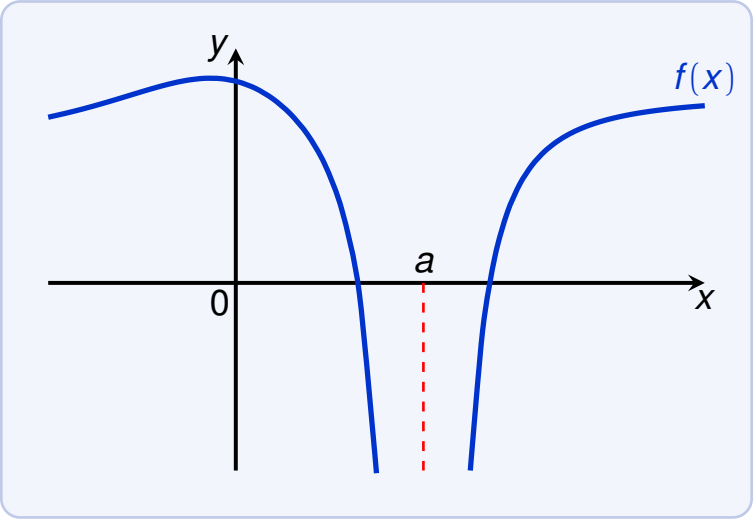
Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \to a} f(x) = \infty$$

spoken: "the limit of $f(x)$, as x approaches a, is infinity"

if we can make the values of $f(x)$ arbitrarily large by taking x to be sufficiently close to a (but not equal to a).

Infinite Limits: Definition



Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \to a} f(x) = -\infty$$

spoken: "the limit of $f(x)$, as x approaches a, is negative infinity"

if we can make the values of $f(x)$ arbitrarily large negative by taking x to be sufficiently close to a (but not equal to a).

Infinite One-Sided Limits

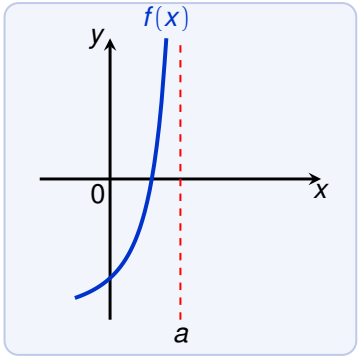
Like wise we define the one-sided infinite limits:

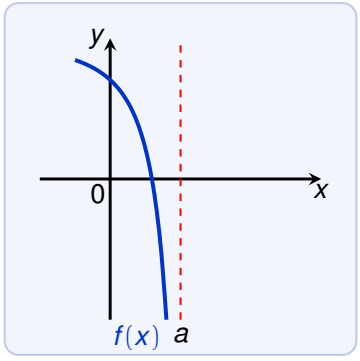
(a) $\lim_{x \to a^-} f(x) = \infty$

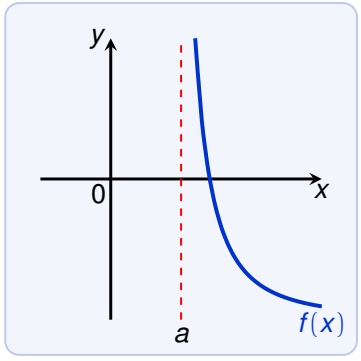
(b) $\lim_{x \to a^-} f(x) = -\infty$

(c) $\lim_{x \to a^+} f(x) = \infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

(a)


(b)


(c)


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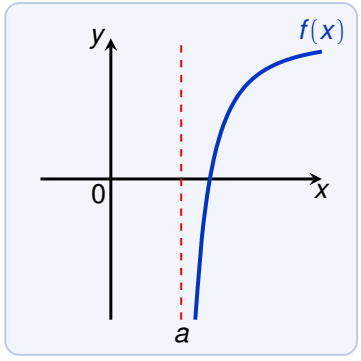

Infinite One-Sided Limits

Like wise we define the one-sided infinite limits:

(a) $\lim_{x \to a^-} f(x) = \infty$

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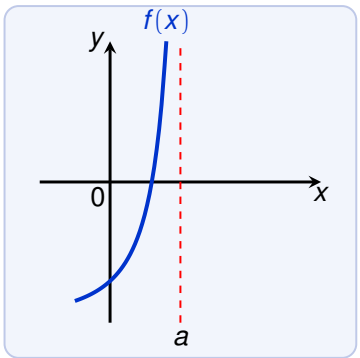
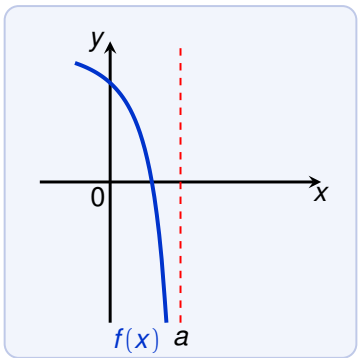


(b)


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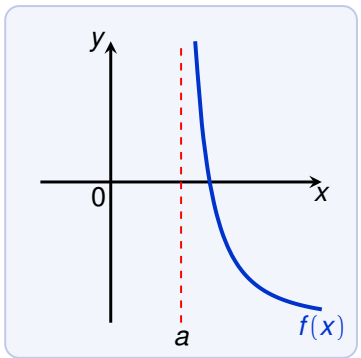

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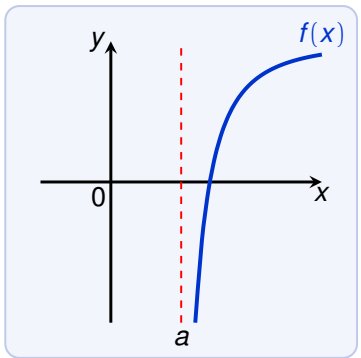

Note that ∞ and $-\infty$ are not considered numbers.

Infinite One-Sided Limits

Like wise we define the one-sided infinite limits:

(a) $\lim_{x \to a^-} f(x) = \infty$

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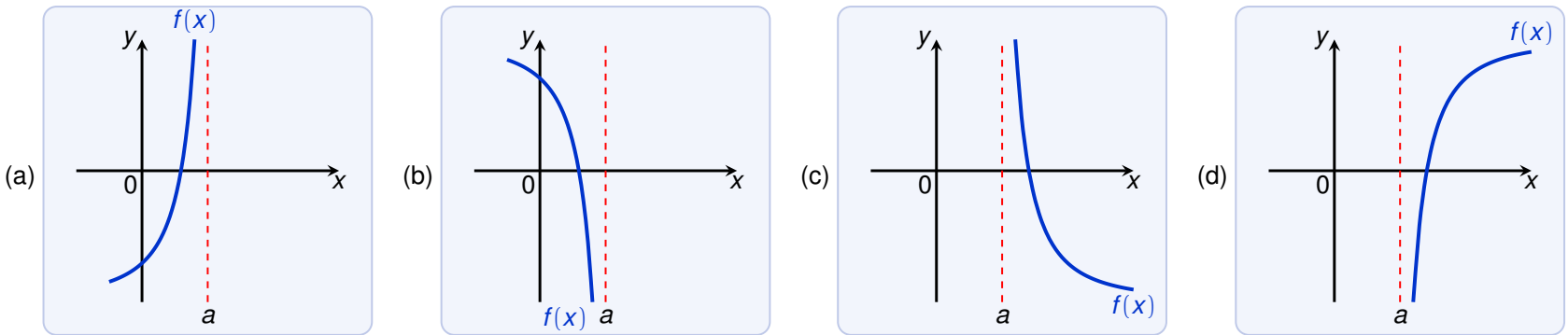

Note that ∞ and $-\infty$ are not considered numbers.

If $\lim_{x \to a} f(x) = \infty$ then $\lim_{x \to a} f(x)$ does not exist.
It indicates a certain way in which the limit does not exist.

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

Infinite Limits: Examples

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$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

$\lim_{x \to 3^-} \frac{2x}{x-3} = ?$ a 0 b 1 c ∞ d $-\infty$

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$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$
$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

If x is close to 3 and $x < 3$ (approaching from the left), then:

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

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$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

If x is close to 3 and $x < 3$ (approaching from the left), then:

- $2x$ is close to 6,

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$

$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$

If x is close to 3 and $x < 3$ (approaching from the left), then:

- $2x$ is close to 6,
- $x - 3$ is a small negative number,

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$

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If x is close to 3 and $x < 3$ (approaching from the left), then:

- $2x$ is close to 6,
- $x - 3$ is a small negative number,
- and thus $2x/(x-3)$ is a large negative number.

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

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Hence $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$.

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- $x - 3$ is a small negative number,
- and thus $2x/(x-3)$ is a large negative number.

Hence $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$.

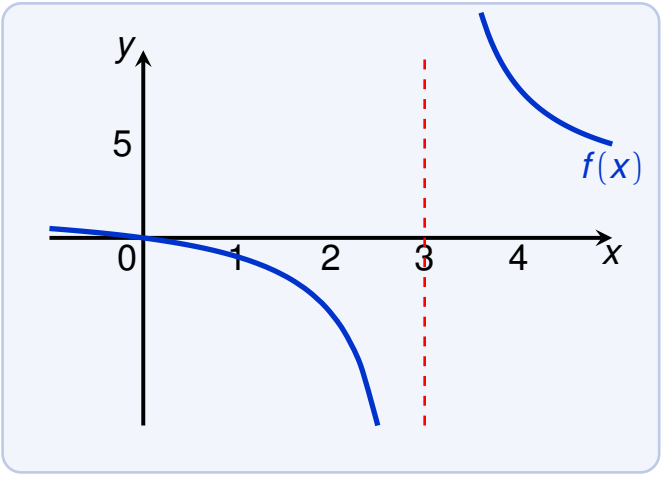
Similarly for x close to 3 and $x > 3$, but now $x - 3$ is positive.

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$
$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$



If x is close to 3 and $x < 3$ (approaching from the left), then:

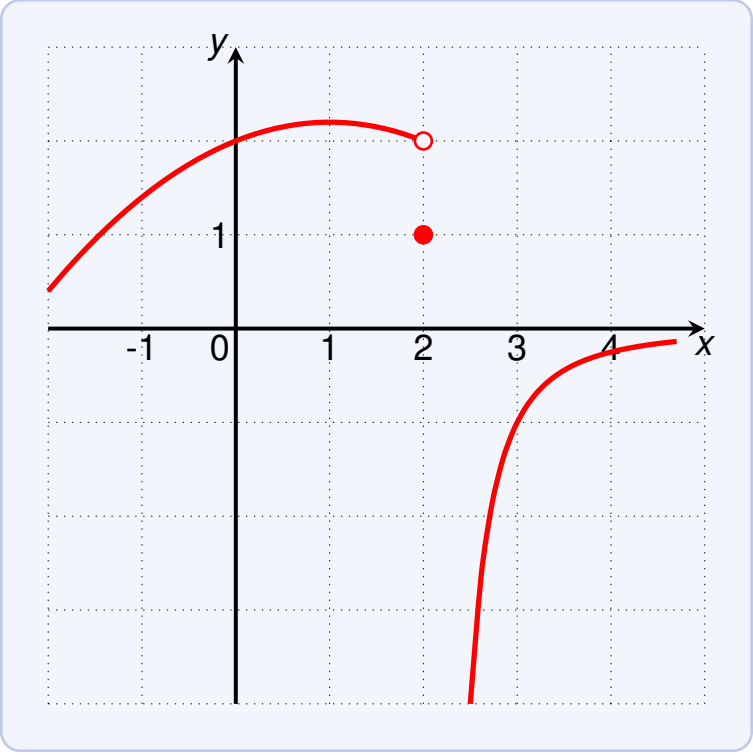
- $2x$ is close to 6,
- $x - 3$ is a small negative number,
- and thus $2x/(x-3)$ is a large negative number.

Hence $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$.

Similarly for x close to 3 and $x > 3$, but now $x - 3$ is positive.

Infinite Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

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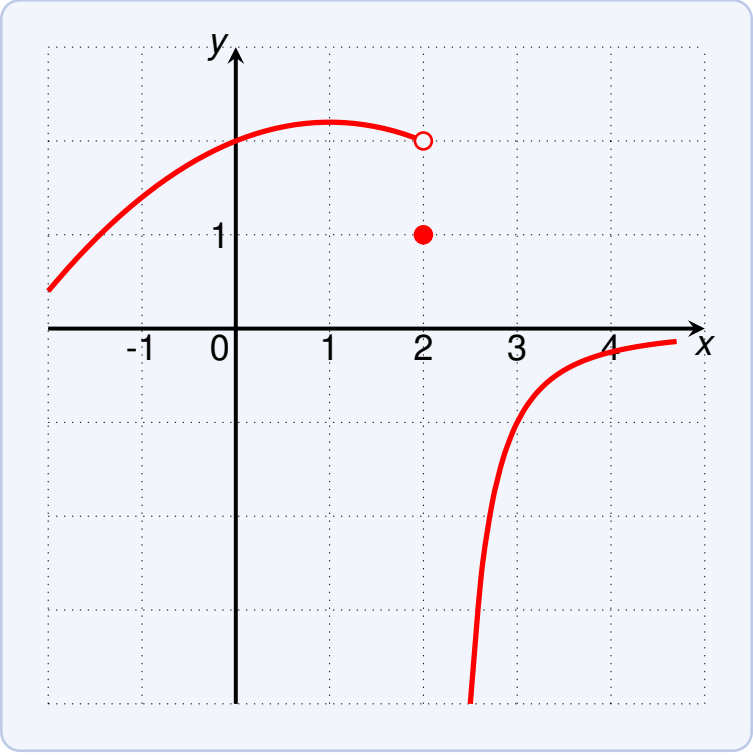

Use the graph to estimate the following values:

- $f(2) = ?$ a 0 b 1 c 2 d undefined

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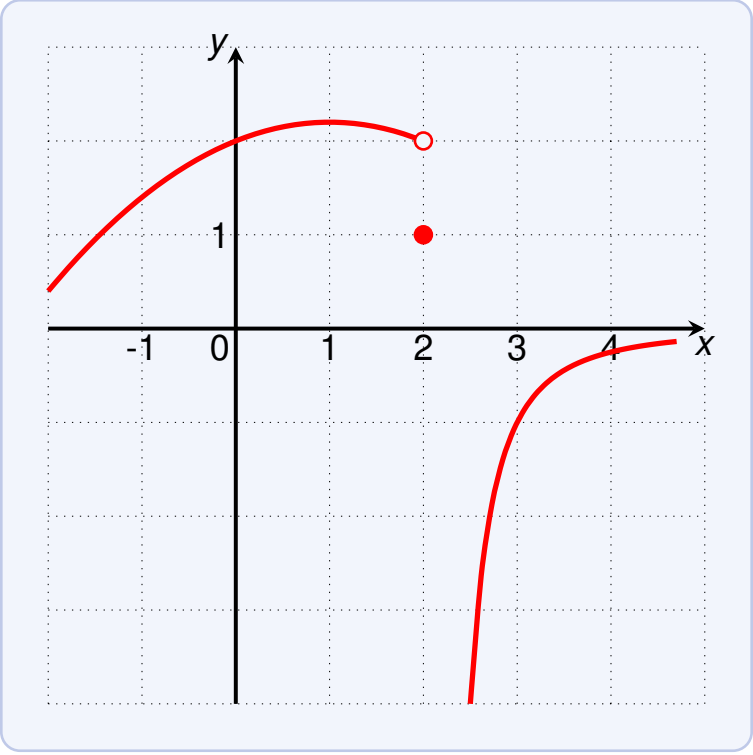

Use the graph to estimate the following values:

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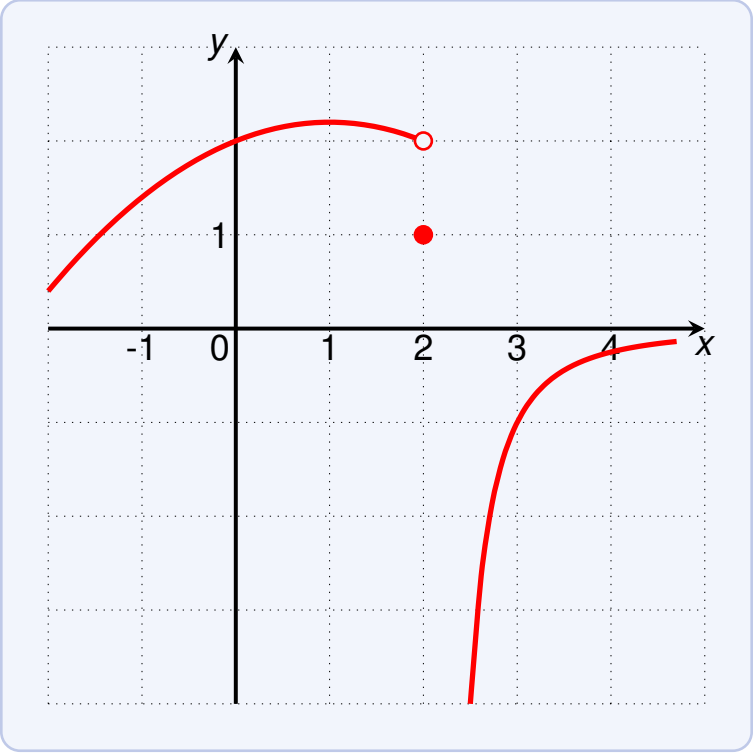

Use the graph to estimate the following values:

- $f(2) = 1$
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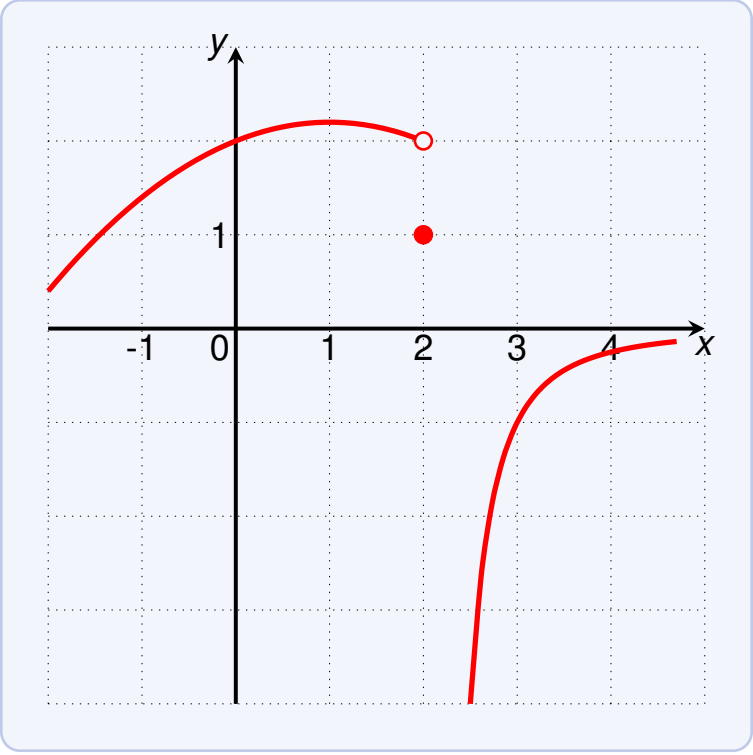

Use the graph to estimate the following values:

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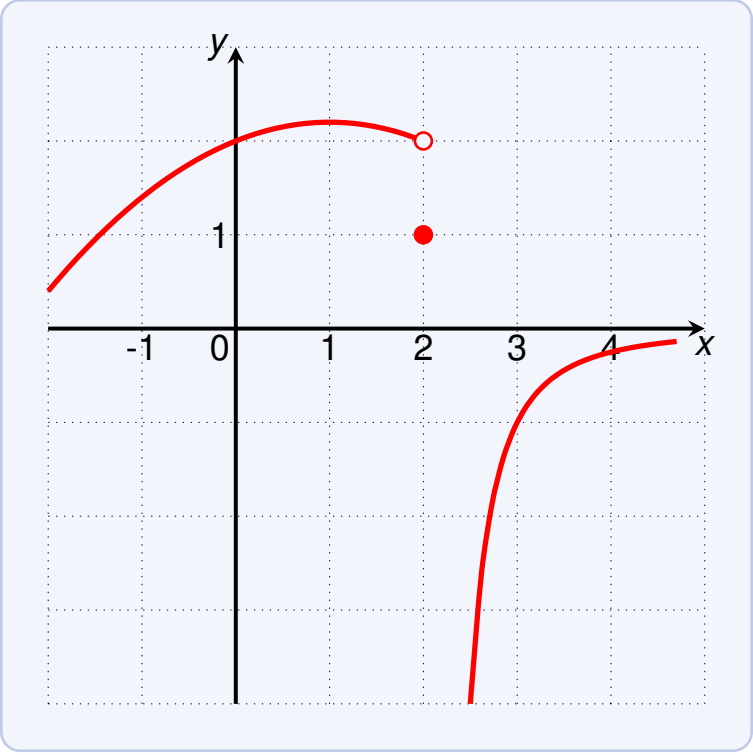

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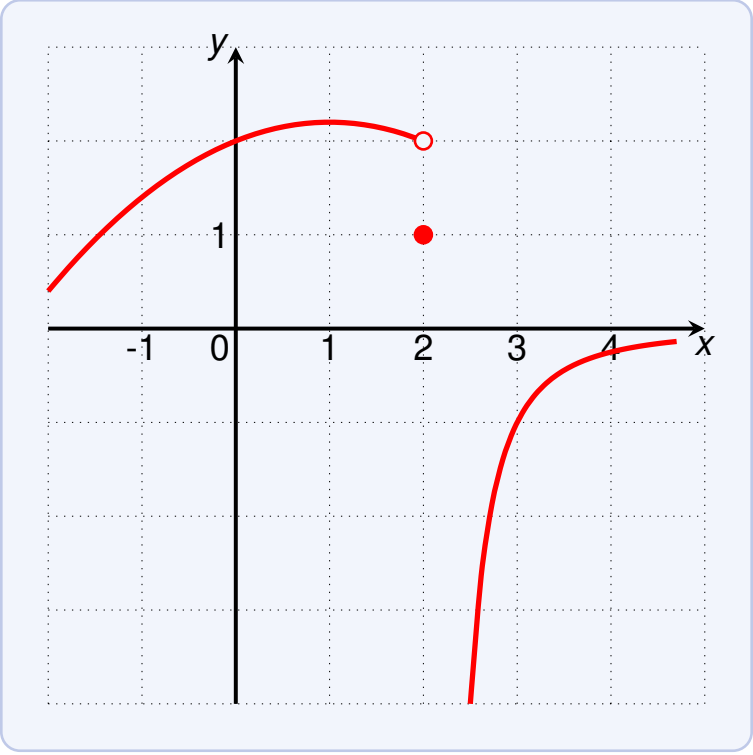

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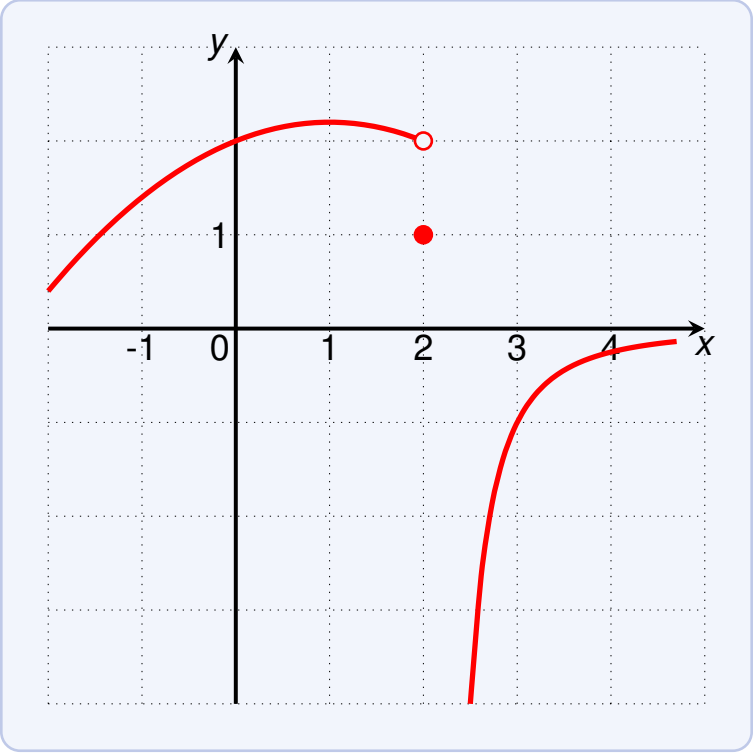

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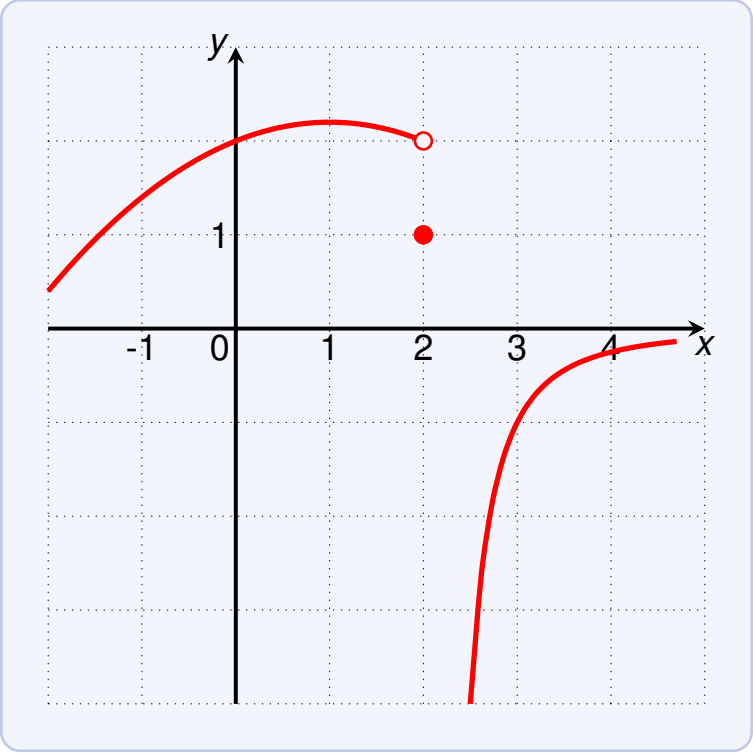


Use the graph to estimate the following values:

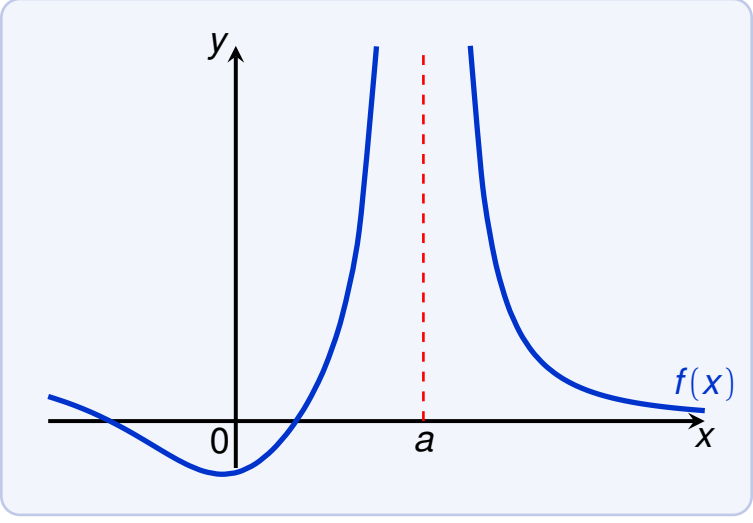
- $f(2) = 1$
- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = -\infty$ (special case of 'does not exist')
- $\lim_{x \to 2}$ does not exist

Infinite Limits: Vertical Asymptotes

The line $x = a$ is a **vertical asymptote** of a function f if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



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What are the vertical asymptotes of

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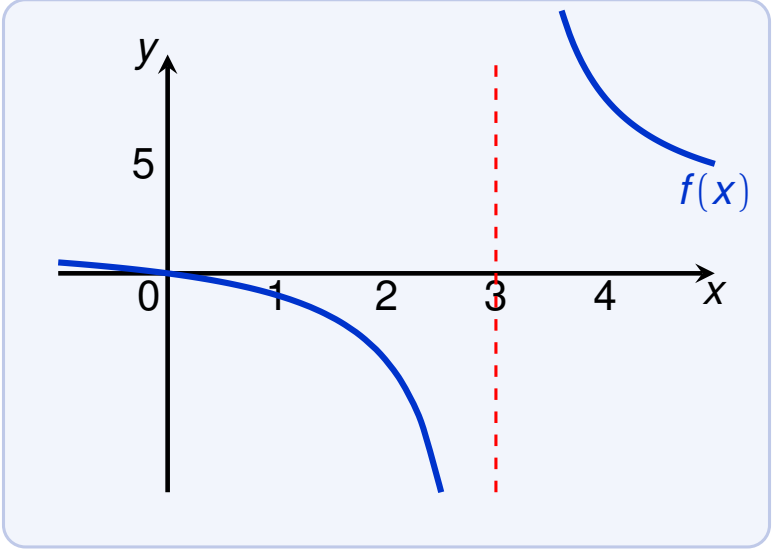

The function has the vertical asymptote $x = 3$:

$$\lim_{x \to 3^-} f(x) = -\infty$$

Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

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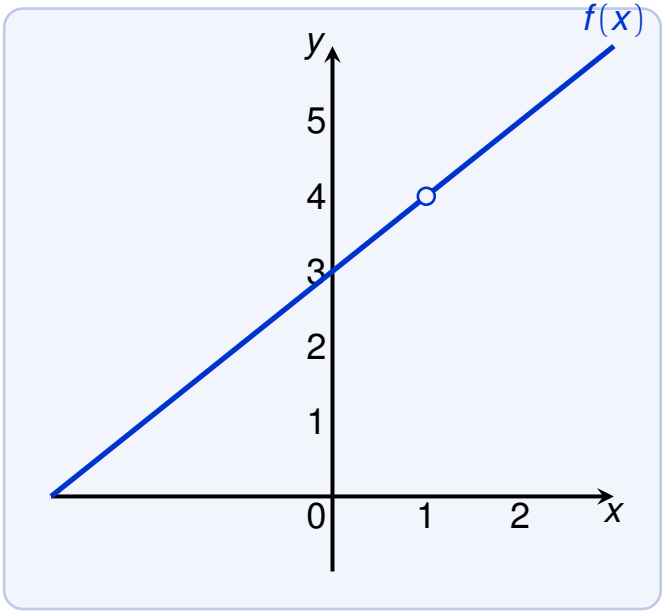

The function has no vertical asymptotes:

$$\frac{x^2 + 2x - 3}{x - 1} = (x + 3) \text{ for } x \neq 1$$

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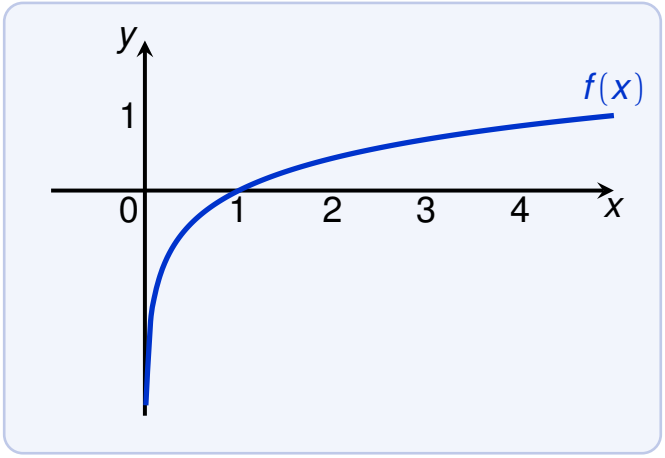

The function has the vertical asymptote $x = 0$:

$$\lim_{x \to 0^+} \log_5 x = -\infty$$