Automata Theory :: Complexity

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Big O Notation

Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$. Then

$$f \in O(g) \quad \iff \quad \exists C > 0.\ \exists n_0.\ f(n) \leq C \cdot g(n) \text{ for all } n \geq n_0$$

$$\begin{aligned} n^a &\in O(n^b) && \text{for all } 0 < a \leq b \\ c_a n^a + c_{a-1} n^{a-1} + \cdots + c_0 &\in O(n^a) && \text{for all } a > 0 \\ n^a &\in O(b^n) && \text{for all } a > 0 \text{ and } b > 1 \\ \log_a n &\in O(n^b) && \text{for all } a, b > 0 \\ \log_a n &\in O(\log_b n) && \text{for all } a, b > 0 \end{aligned}$$

By definition $\log_a a^n = n$. This implies $a^{\log_a n} = n$, and hence

$$a^{\log_a b \cdot \log_b n} = \left(a^{\log_a b}\right)^{\log_b n} = b^{\log_b n} = n$$

Hence $\log_a b \cdot \log_b n = \log_a n$.

Time Complexity: P and NP

Time Complexity

Let $f, g : \mathbb{N} \to \mathbb{N}$.

A nondeterministic Turing machine M

runs in time f

if for every input w, every computation of M reaches a halting state after at most $f(|w|)$ steps.

The function f gives an upper bound on the number of computation steps in terms of the length of the input word.

A Turing machine M has

time complexity $O(g)$

if there exists $f \in O(g)$ such that M runs in time f.

Complexity Classes P and NP

A nondeterministic Turing machine M is **polynomial time** if M runs in time p for some polynomial p.

Equivalently, M has time complexity $O(n^k)$ for some k.

NP is the class of languages accepted by nondeterministic polynomial time Turing machines:

$$\mathbf{NP} = \{\, L(M) \mid M \text{ is nondeterministic polynomial time TM} \,\}$$

P is the class of languages accepted by deterministic polynomial time Turing machines:

$$\mathbf{P} = \{\, L(M) \mid M \text{ is deterministic polynomial time TM} \,\}$$

Clearly P $\subseteq$ NP, but it is unknown whether P $=$ NP.

Problems in NP

Recall, that the language corresponding to a decision problem consists of words representing instances of the problem for which the answer is **yes**.

Intuitively a problem is in NP if:

- every instance has a finite set of possible solutions,
- correctness of a solution can be checked in polynomial time

The question whether the **travelling salesman problem** has a solution of length $\leq k$ is in NP.

Satisfiability in propositional logic is in NP.

The questions if a number is **not prime** is in NP.

Surprisingly, last question in P. (Agrawal, Kayal, Saxena, 2002)

Satisfiability in Propositional Logic

A formula of propositional logic consists of

`true`	conjunction $\wedge$	variables
`false`	disjunction $\vee$	negation $\neg$

A formula of propositional logic ϕ is **satisfiable** if there exists an assignment of `true` and `false` to the variables such that ϕ evaluates to `true`.

Theorem

Satisfiability of formulas of propositional logic is in NP.

Proof.

We can construct a nondeterministic Turing machine that

- guesses an assignment of `true` and `false` to the variables,
- evaluates the formula (in polynomial time), and

accepts if the evaluation is `true`. □

NP-completeness

NP-completeness

Let $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ be decision problems (languages).

Then L_1 is **polynomial-time reducible** to L_2 if there exists a **polynomial-time computable** function $f : \Sigma_1^* \to \Sigma_2^*$ such that:

$$x \in L_1 \iff f(x) \in L_2$$

To decide if $x \in L_1$, we can compute $f(x)$ and check $f(x) \in L_2$.

So the problem L_1 is reduced to the problem L_2.

Let $f : \Sigma_1^* \to \Sigma_2^*$ and $g : \Sigma_2^* \to \Sigma_3^*$ be polynomial-time reductions. The composition $g \circ f : \Sigma_1^* \to \Sigma_3^*$ is a polynomial-time reduction.

NP-completeness

A language $L \in \mathrm{NP}$ is **NP-complete** if every language in NP is polynomial time reducible to L.

Examples of NP-complete Problems

The question whether the **travelling salesman problem** has a solution of length $\leq k$ is NP-complete.

Satisfiability for formulas of propositional logic is NP-complete.

The question whether a graph contains a **Hamiltonian cycle** (a cycle that visits each node exactly once) is NP-complete.

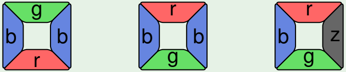
The **bounded tiling problem** is NP-complete.

. . . and many more questions

Bounded Tiling Problem

Bounded Tiling Problem

Given a finite collection of **types** of 1×1 **tiles** with a **colour** on each side. (There are infinitely many tiles of each type.)

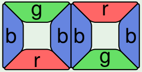


Bonded tiling problem: the input is $n \in \mathbb{N}$, a finite collection of types of tiles, the first row of n tiles.

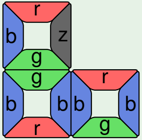
Is it possible to tile an $n \times n$ field (with the given first row)?

When connecting tiles, the touching side must have the same colour. Tiles must not be rotated.

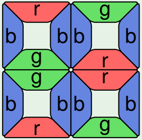
Example $n = 2$:



first row



incomplete tiling



correct tiling

Bounded Tiling Problem is NP-complete

Theorem

The bounded tiling problem is NP-complete.

Proof

First, we argue that the bounded tiling problem is in NP.

We can construct a nondeterministic Turing machien that

- guesses an $n \times n$ tiling, and
- afterwards checks whether the solution is correct.

Both steps can be done in polynomial time.

Second, we show NP-completeness.

Let M be a nondeterministic polynomial-time Turing machine.
Then M has running time $p(k)$ for some polynomial $p(k)$.

We give a polynomial-time reduction of $x \in L(M)\,?$ to the bounded tiling problem.

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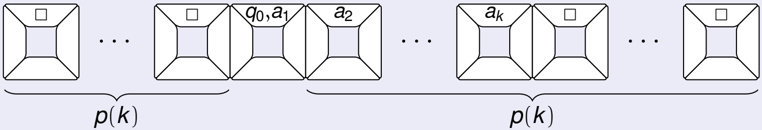
Bounded Tiling Problem is NP-complete

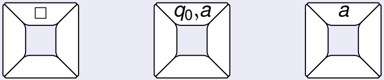
Proof continued... (the starting row)

For input word $x = a_1 \cdots a_k$ we choose $n = 2p(k) + 1$.
(Assume $p(k) \geq k$, otherwise make it so.)

As first row we choose:



Tiles for building the first row (for every $a \in \Sigma$):



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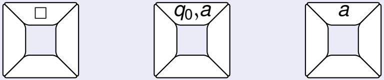
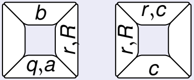
Bounded Tiling Problem is NP-complete

Proof continued… (the types of tiles)

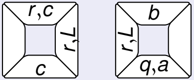
Tiles for building the first row (for every $a \in \Sigma$):



Tiles simulating the computation of M (for every $c \in \Gamma$):

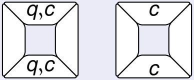


for $(r, b, R) \in \delta(q, a)$



for $(r, b, L) \in \delta(q, a)$

Tiles for leaving the tape unchanged (for every $q \in F$, $c \in \Gamma$):



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Bounded Tiling Problem is NP-complete

Proof continued...

Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

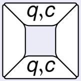
$$n \times n \text{ field can be tiled} \quad \iff \quad x \in L(M)$$

Every tiling simulates a computation of M on input x.

The computation takes at most $p(k)$ steps.

So the computation fills only $p(k) < n$ rows of the tiling.

Hence, the $n \times n$ tiling can only be completed using



which exists only for $q \in F$.

Tiling can be finished

$$\iff M \text{ has an accepting computation for input } x.$$

Example

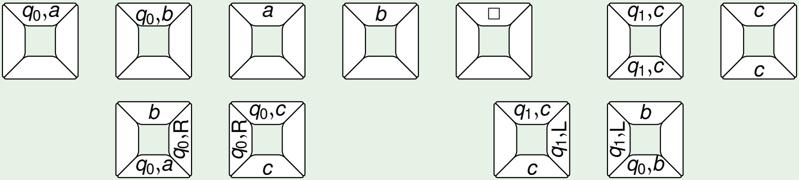
Consider the TM M with $\Sigma = \{a, b\}$, $\Gamma = \Sigma \cup \{\square\}$, $F = \{q_1\}$ and

$$\delta(q_0, a) = \{(q_0, b, R)\} \qquad \delta(q_0, b) = \{(q_1, b, L)\}$$

Note that $L(M) = L(a^* b(a+b)^*) = L((a+b)^* b(a+b)^*)$

For input x, M takes at most $|x|$ steps. So we take $p(k) = k$.

The **tile types** are:



for every $c \in \Gamma$.

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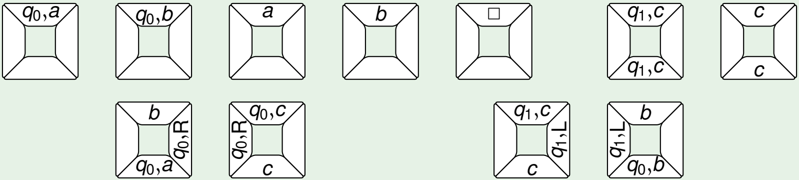
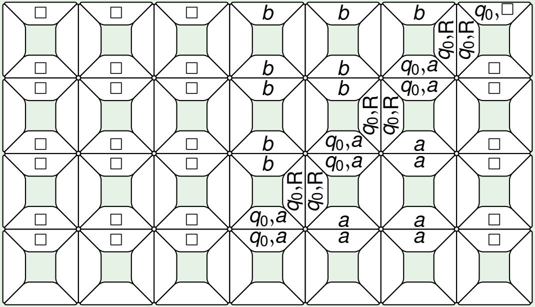
Example



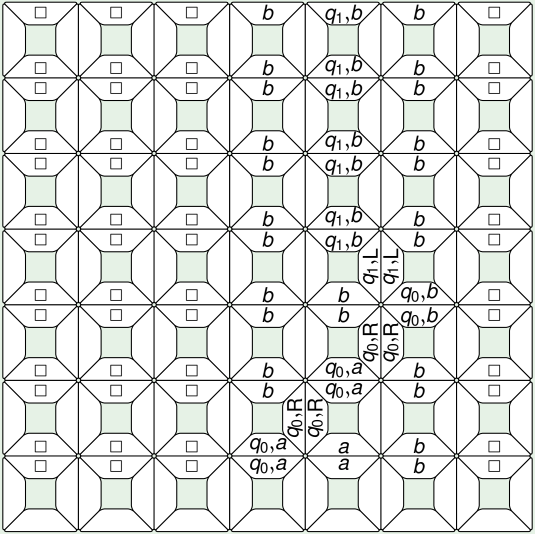
Consider the input word $aaa \notin L(M)$. Then $n = 2p(3) + 1 = 7$.



The tiling cannot be completed.

Example continued

Consider the input word $aab \in L(M)$. Then $n = 2p(3) + 1 = 7$.



Complete tiling of the 7×7 field.

Satisfiability Problem

Satisfiability Problem is NP-complete

Theorem of Cook

The satisfiability problem in propositional logic is NP-complete.

Proof

We give a polynomial-time reduction from the bounded tiling problem to the satisfiability problem.

Given

- a set T of tile types,
- a number n,
- a first row of tiles $t_1 \cdots t_n$.

We create a satisfiability problem as follows.

We introduce Boolean variables x_{rct} for $1 \leq r, c \leq n$ and $t \in T$.

Intention: $x_{rct} = \texttt{true} \iff$ tile of type t at row r and column c.

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Satisfiability Problem is NP-complete

Proof continued...

We define Φ to be the conjunction of the 4 formulas:

1. Fist row is $t_1 \cdots t_n$: $\bigwedge_{c=1}^{n} x_{1ct_k}$
2. At every position at most one tile type:
$$\bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigwedge_{t \neq t'} \neg(x_{rct} \wedge x_{rct'})$$
3. Neighbouring tiles must match (horizontal neighbours):
$$\bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n-1} \bigvee_{tt' \text{ matches}} (x_{rct} \wedge x_{r(c+1)t'})$$
4. Neighbouring tiles must match (vertical neighbours):
$$\bigwedge_{r=1}^{n-1} \bigwedge_{c=1}^{n} \bigvee_{\substack{t' \\ t} \text{ matches}} (x_{rct} \wedge x_{(r+1)ct'})$$

Size of the formula is polynomial in n.

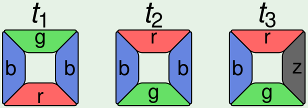
There exists an $n \times n$ tiling with first row $t_1 \cdots t_n$
$\iff$ the propositional formula Φ is satisfiable.

Thus we have a polynomial-time reduction. $\square$

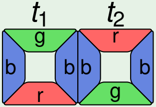
Exercise

Reduce this bounded tiling problem to the satisfiability problem.

Types of tiles:



First row:



Then Φ is the conjunction of:

1. $x_{11t_1} \wedge x_{12t_2}$
2. $\neg(x_{11t_1} \wedge x_{11t_2}) \wedge \neg(x_{12t_1} \wedge x_{12t_2}) \wedge \neg(x_{21t_1} \wedge x_{21t_2}) \wedge \neg(x_{22t_1} \wedge x_{22t_2}) \wedge$
   $\neg(x_{11t_1} \wedge x_{11t_3}) \wedge \neg(x_{12t_1} \wedge x_{12t_3}) \wedge \neg(x_{21t_1} \wedge x_{21t_3}) \wedge \neg(x_{22t_1} \wedge x_{22t_3}) \wedge$
   $\neg(x_{11t_2} \wedge x_{11t_3}) \wedge \neg(x_{12t_2} \wedge x_{12t_3}) \wedge \neg(x_{21t_2} \wedge x_{21t_3}) \wedge \neg(x_{22t_2} \wedge x_{22t_3})$
3. $\big((x_{11t_1} \wedge x_{12t_1}) \vee (x_{11t_1} \wedge x_{12t_2}) \vee (x_{11t_1} \wedge x_{12t_3}) \vee$
   $(x_{11t_2} \wedge x_{12t_1}) \vee (x_{11t_2} \wedge x_{12t_2}) \vee (x_{11t_2} \wedge x_{12t_3})\big) \wedge$
   $\big((x_{21t_1} \wedge x_{22t_1}) \vee (x_{21t_1} \wedge x_{22t_2}) \vee (x_{21t_1} \wedge x_{22t_3}) \vee$
   $(x_{21t_2} \wedge x_{22t_1}) \vee (x_{21t_2} \wedge x_{22t_2}) \vee (x_{21t_2} \wedge x_{22t_3})\big)$
4. $\big((x_{11t_1} \wedge x_{21t_2}) \vee (x_{11t_1} \wedge x_{21t_3}) \vee (x_{11t_2} \wedge x_{21t_1}) \vee (x_{11t_3} \wedge x_{21t_1})\big) \wedge$
   $\big((x_{12t_1} \wedge x_{22t_2}) \vee (x_{12t_1} \wedge x_{22t_3}) \vee (x_{12t_2} \wedge x_{22t_1}) \vee (x_{12t_3} \wedge x_{22t_1})\big)$

$P = NP?$

NP-completeness and P = NP?

Theorem

If an NP-complete language L is also in P, then P $=$ NP.

Proof.

Assume that L is NP-complete and in P.

Let $L' \in$ NP.

As L is NP-complete, there is a polynomial-time reduction f with

$$x \in L' \iff f(x) \in L$$

Since $L \in$ P, we can compute $f(x) \in L$ in polynomial time.

Thus $x \in L'$ can be decided in polynomial time.

Hence $L' \in$ P. $\square$

For proving P $=$ NP it suffices to show that one NP-complete problem can be solved in deterministic polynomial time.

co-NP

The Class co-NP

A problem L is in co-NP if the complement $\overline{L}$ is in NP.

In other words, the set of instances without solution is in NP.

The question whether a propositional formula is **not** satisfiable is in co-NP.

It is unknown whether NP $=$ co-NP.

It is unknown whether the satisfiability problem is in co-NP.

The difficulty is that it has to be shown that a formula evaluates to `false` for **every** variable assignment.

Theorem

If an NP-complete problem is in co-NP, then NP $=$ co-NP.

Note that there are problems that are both in NP $\cap$ co-NP.

Space Complexity

Space Complexity

Let $f, g : \mathbb{N} \to \mathbb{N}$.

A nondeterministic Turing machine M

runs in space f

if for every input w, every computation of M visits at most $f(|w|)$ positions on the tape.

The function f gives an upper bound on the number of visited cells on the tape in terms of the length of the input word.

Complexity Classes PSpace and NPSpace

A nondeterministic Turing machine M is **polynomial space** if M runs in space p for some polynomial p.

$$\textbf{NPSpace} = \{\, L(M) \mid M \text{ nondeterministic polynomial space TM} \,\}$$
$$\textbf{PSpace} = \{\, L(M) \mid M \text{ deterministic polynomial space TM} \,\}$$

$$\text{P} \subseteq \text{PSpace} \qquad\qquad \text{NP} \subseteq \text{NPSpace}$$

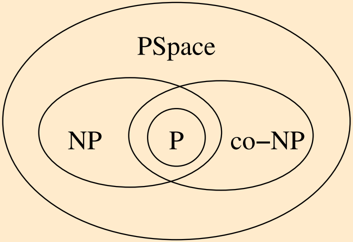
Theorem of Savitch

$$\text{PSpace} = \text{NPSpace}$$

Actually, the theorem says something more general:

If L is accepted by a nondeterministic TM in $f(n)$ space, then L is accepted by a deterministic TM in $f(n)^2$ space.

PSpace-completeness



It is unknown whether these inclusions are strict.

A language $L \in$ PSpace is **PSpace-complete** if every language $L' \in$ PSpace is polynomial-time reducible to L.

$L(r) = \Sigma^*$? for regular expression r is PSpace-complete.

The Classes EXP, NEXP and EXPSpace

The Classes EXP and NEXP

A nondeterministic Turing machine M is

- **exponential time** if M runs in time $2^{p(|x|)}$ and
- **exponential space** if M runs in space $2^{p(|x|)}$

for some polynomial p.

$$\begin{aligned}
\textbf{NEXP} &= \{\, L(M) \mid M \text{ nondeterm. exponential time TM} \,\} \\
\textbf{EXP} &= \{\, L(M) \mid M \text{ deterministic exponential time TM} \,\} \\
\textbf{NEXPSpace} &= \{\, L(M) \mid M \text{ nondeterm. exponential space TM} \,\} \\
\textbf{EXPSpace} &= \{\, L(M) \mid M \text{ deterministic exponential space TM} \,\}
\end{aligned}$$

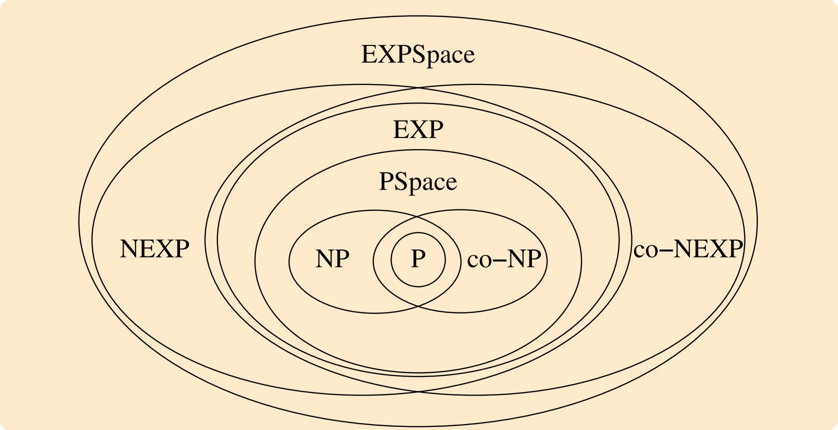
$$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSpace} \subseteq \mathrm{EXP} \subseteq \mathrm{NEXP} \subseteq \mathrm{EXPSpace}$$

It is unknown whether these inclusions are strict. We know

$$\mathrm{P} \neq \mathrm{EXP} \quad \mathrm{NP} \neq \mathrm{NEXP} \quad \mathrm{PSpace} \neq \mathrm{EXPSpace} = \mathrm{NEXPSpace}$$

PSpace $\subseteq$ EXP holds since a polynomial-space TM can at most take an exponential number of configurations.

Complexity Hierarchy



The following inclusions are known to be strict:

$$P \neq EXP \qquad NP \neq NEXP \qquad PSpace \neq EXPSpace$$