Automata Theory :: Unrestricted Grammars

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Theorem

A language L is generated by an unrestricted grammar $\iff L$ is accepted by a Turing machine.

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For every unrestricted ${\it G}$ there is a Turing machine ${\it M}$ such that

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The variables are S, T, \square and $V_{\gamma}^{\alpha}, V_{q\gamma}^{\alpha}$ for every $\alpha \in \Sigma \cup \{\square\}, \gamma \in \Gamma$ and $q \in Q$.

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Step 1: guessing the word w

$$S
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 $T
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for every $a \in \Sigma$

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 $T o TV_a^a \mid V_{q_0a}^a$ for every $a\in \Sigma$

After step 1, we have derived something of the form

$$V_{\square}^{\square}\cdots V_{\square}^{\square}V_{q_0a_1}^{a_1}V_{a_2}^{a_2}V_{a_3}^{a_3}\cdots V_{a_n}^{a_n}V_{\square}^{\square}\cdots V_{\square}^{\square}$$

where $w = a_1 a_2 \cdots a_n$.

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where $w = a_1 a_2 \cdots a_n$.

Next, the TM is simulated using the lower line (the subscripts).

Construction continued

Step 2: simulating the TM (in the subscripts)

$$V_{qc}^{\alpha}V_{\gamma}^{\beta} \rightarrow V_{d}^{\alpha}V_{q'\gamma}^{\beta}$$
 if $\delta(q,c) = (q',d,R)$
 $V_{\gamma}^{\beta}V_{qc}^{\alpha} \rightarrow V_{q'\gamma}^{\beta}V_{d}^{\alpha}$ if $\delta(q,c) = (q',d,L)$

for every $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

Construction continued

Step 2: simulating the TM (in the subscripts)

$$egin{aligned} V^{lpha}_{m{qc}} V^{eta}_{m{\gamma}} &
ightarrow V^{lpha}_{m{d}} V^{eta}_{m{q'\gamma}} & ext{if } \delta(m{q},m{c}) = (m{q'},m{d},m{R}) \ V^{eta}_{m{\gamma}} V^{lpha}_{m{qc}} &
ightarrow V^{eta}_{m{q'\gamma}} V^{lpha}_{m{d}} & ext{if } \delta(m{q},m{c}) = (m{q'},m{d},m{L}) \end{aligned}$$

for every $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

Step 3: If TM reaches accepting state, then generate *w*. (From the superscripts left unchanged in step 2.)

$$egin{aligned} V_{q\gamma}^{lpha} & \to & lpha \ eta V_{\gamma}^{lpha} & \to & eta lpha \ V_{\gamma}^{lpha} & eta & au eta \ & \Box & \to & \lambda \end{aligned} \qquad ext{for every } q \in F$$

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 $V^{\beta}_{\gamma}V^{\alpha}_{qc} \rightarrow V^{\beta}_{q'\gamma}V^{\alpha}_{d}$ if $\delta(q,c) = (q',d,L)$

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Then L(G) = L(M).

Consider the TM with $\Sigma=\{a,b,c\},\ \Gamma=\Sigma\cup\{\Box\},\ F=\{q_2\}$ and $\delta(q_0,a)=(q_0,c,R) \qquad \qquad \delta(q_0,c)=(q_1,b,L)$ $\delta(q_0,b)=(q_0,b,R) \qquad \qquad \delta(q_1,b)=(q_2,a,R)$

This TM accepts the language L((a+b)*bc(a+b+c)*).

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 $V_{q_{0}a}^{\alpha} V_{\gamma}^{\beta}
ightarrow V_{c}^{\alpha} V_{q_{0}\gamma}^{\beta} \qquad V_{q_{2}\gamma}^{\alpha}
ightarrow \alpha$
 $V_{q_{0}b}^{\alpha} V_{\gamma}^{\beta}
ightarrow V_{b}^{\alpha} V_{q_{0}\gamma}^{\beta} \qquad \beta V_{\gamma}^{\alpha}
ightarrow \beta \alpha$
 $V_{\gamma}^{\beta} V_{q_{0}c}^{\alpha}
ightarrow V_{q_{1}\gamma}^{\beta} V_{b}^{\alpha} \qquad V_{\gamma}^{\alpha} \beta
ightarrow \alpha \beta$
 $V_{q_{1}b}^{\alpha} V_{\gamma}^{\beta}
ightarrow V_{a}^{\alpha} V_{q_{2}\gamma}^{\beta} \qquad \square
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with α , $\beta \in \Sigma \cup \{\square\}$ and $\gamma \in \Gamma$.

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 $T o TV_a^a \mid TV_b^b \mid TV_c^c \mid V_{q_0a}^a \mid V_{q_0b}^b \mid V_{q_0c}^c$
 $V_{q_0a}^{lpha} V_{\gamma}^{eta} o V_c^{lpha} V_{q_0\gamma}^{eta} o V_{q_2\gamma}^{lpha} o lpha$
 $V_{q_0b}^{lpha} V_{\gamma}^{eta} o V_b^{lpha} V_{q_0\gamma}^{eta} o \beta lpha$
 $V_{\gamma}^{eta} V_{q_0c}^{lpha} o V_{q_1\gamma}^{eta} V_b^{lpha} o V_{\gamma}^{lpha} \beta o lpha$
 $V_{q_1b}^{lpha} V_{\gamma}^{eta} o V_a^{lpha} V_{q_2\gamma}^{eta} o \lambda$

with $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$. Exercise: derive *abc*.