Automata Theory ::
Recursively Enumerable Languages

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such that $L = \{ w_1, w_2, w_3, \ldots \}$.

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If L is infinite, this computation never stops! Every word from L will eventually be written on the tape.

Then $w_1, w_2, \dots$ is called a **recursive enumeration** of L.

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Thus, there is a recursive enumeration of all Turing machines:

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Formally, we enumerate words describing Turing machines. But the does not matter since we have a universal Turing machine that can execute these descriptions.

Properties of Recursively Enumerable Languages

Union and Intersection

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If L_1 and L_2 are recursively enumerable languages, then so are

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(Alternating simulate one step of M_1 and one step of M_2.)

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What about $L_1 \setminus L_2$?

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There exist recursively enumerable languages L, for which the complement $\overline{L}$ is not recursively enumerable.

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Let $M_1, M_2, M_3, \ldots$ be a recursive enumeration of all TMs.

Define the language L by

$$L = \{\, a^i \mid a^i \in L(M_i), i \geq 1 \,\}$$

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Then L is recursively enumerable.

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$$a^k \in \overline{L} \iff a^k \in L(M_k)$$

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$L_1 \setminus L_2$ is not always recursively enumerable since $\overline{L} = \Sigma^* \setminus L$.

Concatenation

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A split of a word w is a pair (w_1, w_2) such that $w = w_1 w_2$.

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We call the split good if M_1 accepts w_1 and M_2 accepts w_2.

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Create a Turing machine N that

- computes all splits of the input word w
- checks **all splits in parallel** whether they are good
- accepts the input w as soon as a good split is found

Then $L(N) = L_1 L_2$.

Kleene Start

Theorem

If L is a recursively enumerable language, then so is

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Let M be a Turing machine such that $L(M) = L$.

A partitioning of a word $w \neq \lambda$ are non-empty words $(w_1, \ldots, w_n)$ for some $n \leq |w|$ such that $w = w_1 w_2 \cdots w_n$.

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Create a Turing machine N that

- computes all partitionings of the input word $w \neq \lambda$
- checks **all partitionings in parallel** whether they are good
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Then $L(N) = L^*$.

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Not every recursively enumerable language is recursive!

(See the a few slides back.)

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Theorem

A language L is recursive $\iff$ L is accepted by a deterministic TM M that reaches for every input a halting state.

Proof on the next slide.

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We show that $\overline{L}$ is recursively enumerable.

From M we construct a Turing machine N as follows:

- Add a fresh state q_f.
- For all $q \in Q \setminus F$ and $a \in \Gamma$:
  if $\delta(q, a)$ undefined, define $\delta(q, a) = (q_f, a, R)$.
- Make q_f the only final state.

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$(\Rightarrow)$ L and $\overline{L}$ accepted by deterministic TMs M_1 and M_2.

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Construct a TM M executes M_1 and M_2 in parallel:

- M accepts when M_1 accepts
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Construct a TM M executes M_1 and M_2 in parallel:

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Then $L(M) = L(M_1) = L$, and M halts for every input.

Properties of Recursive Languages

Theorem

If L, L_1 and L_2 are recursive, then so are

$$\overline{L} \qquad L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1 \setminus L_2 \qquad L^* \qquad L_1 L_2$$

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- $\overline{L}$: $\overline{L}$ and $\overline{\overline{L}} = L$ are r.e.

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- $\overline{L}$: $\overline{L}$ and $\overline{\overline{L}} = L$ are r.e.
- $L_1 \cup L_2$: $L_1 \cup L_2$ and $\overline{L_1 \cup L_2} = \overline{L_1} \cap \overline{L_2}$ are r.e.

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- $L_1 \cap L_2$: $L_1 \cap L_2$ and $\overline{L_1 \cap L_2} = \overline{L_1} \cup \overline{L_2}$ are r.e.

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- $L_1 \cap L_2$: $L_1 \cap L_2$ and $\overline{L_1 \cap L_2} = \overline{L_1} \cup \overline{L_2}$ are r.e.
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$

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Proof.

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- $L_1 \cap L_2$: $L_1 \cap L_2$ and $\overline{L_1 \cap L_2} = \overline{L_1} \cup \overline{L_2}$ are r.e.
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$
- $L_1 L_2$, L^*: same proof as for recursively enumerable languages. Observe that the constructed Turing machine halts on all inputs if M_1, M_2 and M do.

$\square$