Automata Theory :: Pumping Lemma for Context-Free Languages

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Pumping Lemma for Context-Free Languages (1961)

Theorem

Let L be a context-free language.

There exists $m > 0$ such that for every word $w \in L$ with $|w| \geq m$:

$$w = uvxyz$$

with $|vxy| \leq m$ and $|vy| \geq 1$, and $uv^ixy^iz \in L$ for every $i \geq 0$.

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Let G be a context-free grammar with $L(G) = L \setminus \{\lambda\}$

- with k variables, and
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continued on next slide...

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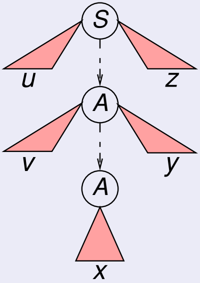

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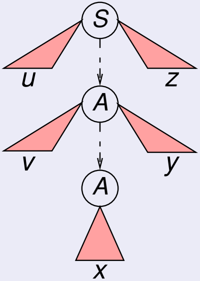

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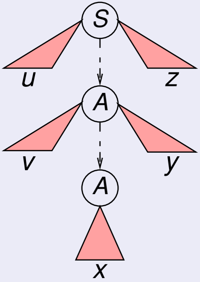

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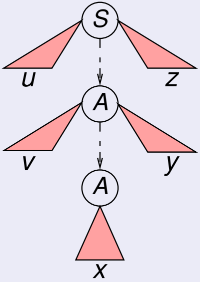

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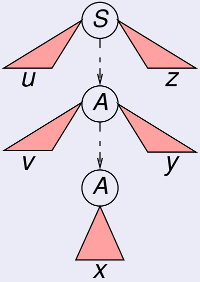

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Then

- $|vxy| \leq m$ as the subtree generating vxy has depth $\leq k + 2$,
- $|vy| \geq 1$, since there are no λ and unit productions. □

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$$\quad\quad \exists u, v, x, y, z \text{ with } w = uvxyz,\ |vxy| \leq m,\ |vy| \geq 1.$$
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If we can always win, then L has no pumping property!

Example

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Intuitively:

- opponent picks m,
- we pick $w = a^m b^m c^m$,
- opponent u, v, x, y, z

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Show that L also has the pumping property for $m = 2$.

Hint: distinguish w of even and odd length when splitting.

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- If $|u| < m$, then $uv^0 xy^0 z = a^{m-j} b^{m-k} a^m b^m \notin L$.

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- If $|u| < m$, then $uv^0 xy^0 z = a^{m-j} b^{m-k} a^m b^m \notin L$.
- If $m \leq |u| < 2m$, then $uv^0 xy^0 z = a^m b^{m-k} a^{m-j} b^m \notin L$.

Exercises (3)

Show that $L = \{ ww \mid w \in \{a,b\}^* \}$ is not context-free.

Assume that L was context-free.

According to the pumping lemma there is $m > 0$ such that

$$a^m b^m a^m b^m = uvxyz$$

with $|vxy| \leq m$, $|vy| \geq 1$, and $uv^i xy^i z \in L$ for every $i \geq 0$.

Since $|vxy| \leq m$, $vy = a^j b^k$ or $vy = b^k a^j$ for some $j, k \geq 0$.

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- If $2m \leq |u|$, then $uv^0 xy^0 z = a^m b^m a^{m-j} b^{m-k} \notin L$.

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Contradiction in each case! Thus L is not context-free.