Automata Theory :: Context-Free Languages

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$$A \to u$$

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Context-free grammars are used to

- describe the **syntax of programming languages**, and
- form the basis of **parsing algorithms**.

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The class of context-free languages is not **closed under complement**.

Derivation Trees and Ambiguity

Rightmost and Leftmost

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- **leftmost**: always the leftmost variable
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Two derivations of *bab*:

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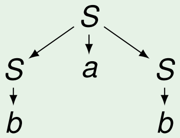
Result depends not on the strategy, but the choice of the rules.

Derivation Trees

A **derivation tree** for a context-free grammar $G = (V, T, S, P)$:

- Nodes have labels from $V \cup T \cup \{\lambda\}$. The **root** has label S.
- A node with label $A \in V$ can have children labelled
  - $x_1, \ldots, x_n$ if there is a rule $A \to x_1 \cdots x_n$ with $n \geq 1$, or
  - λ (single child) if there is a rule $A \to \lambda$.
- Every node with a label from $T \cup \{\lambda\}$ is a **leave.**

In the grammar $S \to SaS \mid b$, a **derivation tree** for *bab* is:



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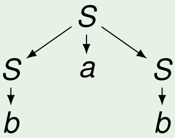

The labels of the leaves of a derivation tree, read from left to right (skipping λ), form a word in $L(G)$.

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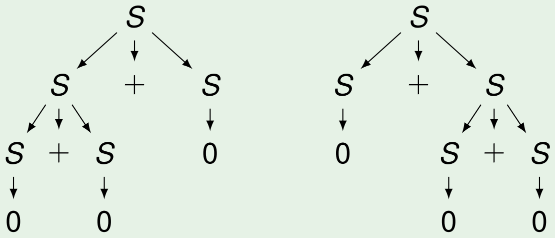

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Note that both grammars

$$S \to S + S \mid 0 \qquad\qquad S \to S + 0 \mid 0$$

generate the same language:

$$\{\, 0(+0)^n \mid n \geq 0 \,\}$$

Ambiguity in Programming Languages

The following production rules (from ALGOL 60) became known as **dangling else problem**:

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The production rules are ambiguous.

Donald Knuth, *The Remaining Trouble Spots in ALGOL 60*, CACM, 1967

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Ambiguity is undecidable. That is, there exists no algorithm that decides whether a context-free grammar is ambiguous.

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It is generated by the following (ambiguous) grammar

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Exercise

Find two derivation trees for the word *abc*.

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For every context-free grammar G:

- there exists an algorithm and $C \geq 0$ such that
- for every word w, the algorithm determines in at most

$$C \cdot |w|^3 \text{ steps}$$

whether $w \in L(G)$ and a derivation tree if $w \in L(G)$.