Automata Theory :: Pumping Lemma

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Non-Regular Languages

Theorem

$L = \{ a^n b^n \mid n \geq 0 \}$ is **not** regular.

Proof.

For a contradiction, assume that L was regular.

Then there exists a DFA $M = (Q, \{a, b\}, \delta, q_0, F)$ with $L(M) = L$.

Because Q is finite, we have

$$q_0 \xrightarrow{a^k}\!\!\!\!\!\twoheadrightarrow q \xrightarrow{a^\ell}\!\!\!\!\!\twoheadrightarrow q$$

for some $\ell \geq 1$ and $q \in Q$. Then for some $q' \in Q$

$$q_0 \xrightarrow{a^k}\!\!\!\!\!\twoheadrightarrow q \xrightarrow{b^k}\!\!\!\!\!\twoheadrightarrow q' \qquad q_0 \xrightarrow{a^k}\!\!\!\!\!\twoheadrightarrow q \xrightarrow{a^\ell}\!\!\!\!\!\twoheadrightarrow q \xrightarrow{b^k}\!\!\!\!\!\twoheadrightarrow q'$$

Then $q' \in F$ since $a^k b^k \in L$. However, $q' \notin F$ since $a^{k+\ell} b^k \notin L$.

Contradiction, thus L is not regular. □

We generalise the idea of the proof. . .

Pumping Lemma for Regular Languages (1959)

Pumping Lemma

Let L be a regular language. There exists $m > 0$ such that every $w \in L$ with $|w| \geq m$ can be written in the form

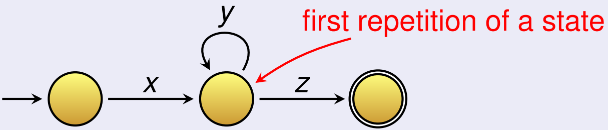
$$w = xyz$$

with $|xy| \leq m$ and $|y| \geq 1$, and $xy^iz \in L$ for every $i \geq 0$.

Proof.

We have $L = L(M)$ for some DFA M with m states.

When M reads $w \in L$ with $|w| \geq m$, there must be a cycle



with $|xy| \leq m$ and $|y| \geq 1$. Then $xy^iz \in L$ for every $i \geq 0$. □

Pumping Lemma Example

The **pumping lemma** can be used to prove that a language is **not regular**.

Assume that $L = \{ w \in \{a, b\}^* \mid w = w^R \}$ is regular.

By the pumping lemma there exists $m > 0$ such that

$$a^m b a^m = xyz$$

with $|xy| \leq m$, $|y| \geq 1$, and $xy^i z \in L$ for every $i \geq 0$.

Since $|xy| \leq m$ and $|y| \geq 1$, it follows that

$$x = a^j \qquad \text{and} \qquad y = a^k$$

with $j \geq 0$ and $k \geq 1$.

However $xyyz = a^{m+k} b a^m \notin L$. Contradiction!

Thus L is not regular. □

Using the Pumping Lemma

Attention

A contradiction for specific m, x, y, or z is not sufficient!

Pumping property as formula (note the quantifiers):

$$\exists m > 0.$$
$$\forall w \in L \text{ with } |w| \geq m.$$
$$\exists x, y, z \text{ with } w = xyz,\ |xy| \leq m,\ |y| \geq 1.$$
$$\forall i \geq 0.\, xy^i z \in L$$

To **contradict the pumping property**, we prove the negation:

$$\forall m > 0.$$
$$\exists w \in L \text{ with } |w| \geq m.$$
$$\forall x, y, z \text{ with } w = xyz,\ |xy| \leq m,\ |y| \geq 1.$$
$$\exists i \geq 0.\, xy^i z \notin L$$

Pumping Lemma as a Game

To **contradict the pumping property**, we prove the negation:

$$\forall m > 0.$$
$$\exists w \in L \text{ with } |w| \geq m.$$
$$\forall x, y, z \text{ with } w = xyz,\ |xy| \leq m,\ |y| \geq 1.$$
$$\exists i \geq 0.\ xy^i z \notin L$$

Pumping Lemma as a Game

Given is a language L. We want to prove that L is not regular.

1. Opponent picks m.
2. We choose a word $w \in L$ with $|w| \geq m$.
3. Opponent picks x, y, z with $w = xyz$, $|xy| \leq m$ and $|y| \geq 1$.
4. If we can find $i \geq 0$ such that $xy^i z \notin L$, then **we win.**

If we can always win, L does not have the pumping property!

Who wins the game when L is finite?

Exercise (1)

Use the pumping lemma to **show that**

$$L = \{ a^n b^n \mid n \geq 0 \}$$

is not regular. Assume that L was regular.

By the pumping lemma there exists $m > 0$ such that

$$a^m b^m = xyz$$

with $|xy| \leq m$, $|y| \geq 1$, and $xy^i z \in L$ for every $i \geq 0$.

Since $|xy| \leq m$ and $|y| \geq 1$, it follows that

$$x = a^j \qquad \text{and} \qquad y = a^k$$

with $j \geq 0$ and $k \geq 1$.

However $xy^0 z = xz = a^{m-k} b^m \notin L$. Contradiction!

Thus L is not regular. □

Exercise (2)

Use the pumping lemma to **show that**

$$L = \{ a^{2^k} \mid k \geq 0 \}$$

is not regular. Assume that L was regular.

By the pumping lemma there exists $m > 0$ such that

$$a^{2^m} = xyz$$

with $|xy| \leq m$, $|y| \geq 1$, and $xy^iz \in L$ for every $i \geq 0$.

Since $|xy| \leq m$ and $|y| \geq 1$, it follows that

$$x = a^j \qquad \text{and} \qquad y = a^k$$

with $j \geq 0$, $k \geq 1$ and $j + k \leq m$.

We have $k \leq m < 2^m$ and hence $2^m < 2^m + k < 2^{m+1}$.

Contradiction: $xy^2z = a^{2^m+k} \notin L$! Thus L is not regular. □