Automata & Complexity

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Big O Notation

Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$. Then

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c_a n^a + c_{a-1} n^{a-1} + \cdots + c_0 &\in O(n^a) && \text{for all } a > 0 \\
n^a &\in O(b^n) && \text{for all } a > 0 \text{ and } b > 1 \\
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By definition $\log_a a^n = n$. This implies $a^{\log_a n} = n$, and hence

$$a^{\log_a b \cdot \log_b n} = \left(a^{\log_a b}\right)^{\log_b n} = b^{\log_b n} = n$$

Hence $\log_a b \cdot \log_b n = \log_a n$.

Time Complexity

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A nondeterministic Turing machine M

runs in time f

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A Turing machine M has

time complexity $O(g)$

if there exists $f \in O(g)$ such that M runs in time f.

Complexity Classes P and NP

A nondeterministic Turing machine M is **polynomial time** if M runs in time p for some polynomial p.

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Clearly $\mathrm{P} \subseteq \mathrm{NP}$, but it is unknown whether $\mathrm{P} = \mathrm{NP}$.

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Surprisingly, last question in P. (Agrawal, Kayal, Saxena, 2002)

Satisfiability in Propositional Logic

A formula of propositional logic consists of

`true`	conjunction $\wedge$	variables
`false`	disjunction $\vee$	negation $\neg$

A formula of propositional logic ϕ is **satisfiable** if there exists an assignment of `true` and `false` to the variables such that ϕ evaluates to `true`.

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Theorem

Satisfiability of formulas of propositional logic is in NP.

Proof.

We can construct a nondeterministic Turing machine that

- guesses an assignment of `true` and `false` to the variables,
- evaluates the formula (in polynomial time), and

accepts if the evaluation is `true`. □

NP-completeness

Let $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ be decision problems (languages).

Then L_1 is **polynomial-time reducible** to L_2 if there exists a **polynomial-time computable** function $f : \Sigma_1^* \to \Sigma_2^*$ such that:

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Let $f : \Sigma_1^* \to \Sigma_2^*$ and $g : \Sigma_2^* \to \Sigma_3^*$ be polynomial-time reductions.
The composition $g \circ f : \Sigma_1^* \to \Sigma_3^*$ is a polynomial-time reduction.

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NP-completeness

A language $L \in$ NP is **NP-complete** if every language in NP is polynomial time reducible to L.

Examples of NP-complete Problems

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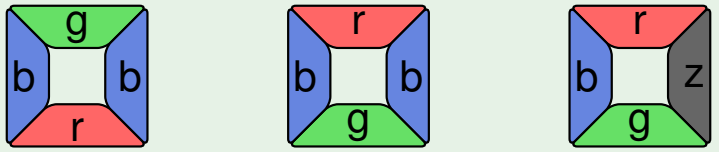
The **bounded tiling problem** is NP-complete.

. . . and many more questions

Bounded Tiling Problem

Given a finite collection of **types** of 1×1 **tiles** with a **colour** on each side. (There are infinitely many tiles of each type.)



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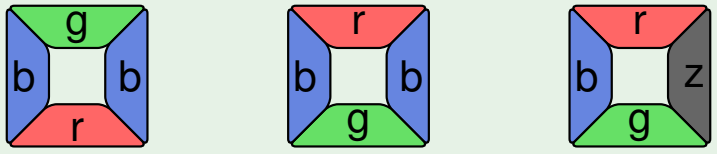

Bonded tiling problem: the input is $n \in \mathbb{N}$, a finite collection of types of tiles, the first row of n tiles.

Is it possible to tile an $n \times n$ field (with the given first row)?

When connecting tiles, the touching side must have the same colour. Tiles must not be rotated.

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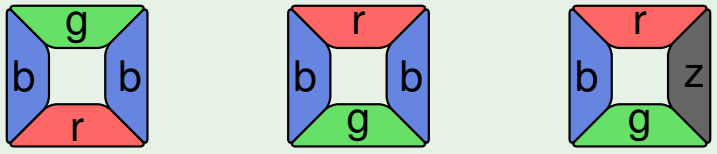

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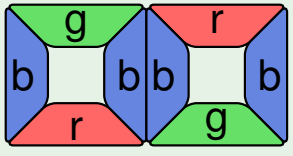
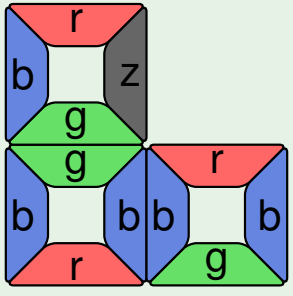
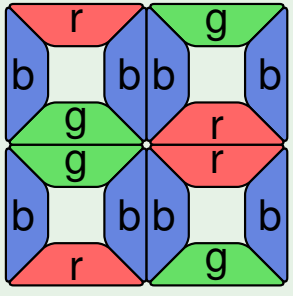
Example $n = 2$:



first row



incomplete tiling



correct tiling

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We can construct a nondeterministic Turing machien that

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continued on the next slide. . .

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Proof continued. . . (the starting row)

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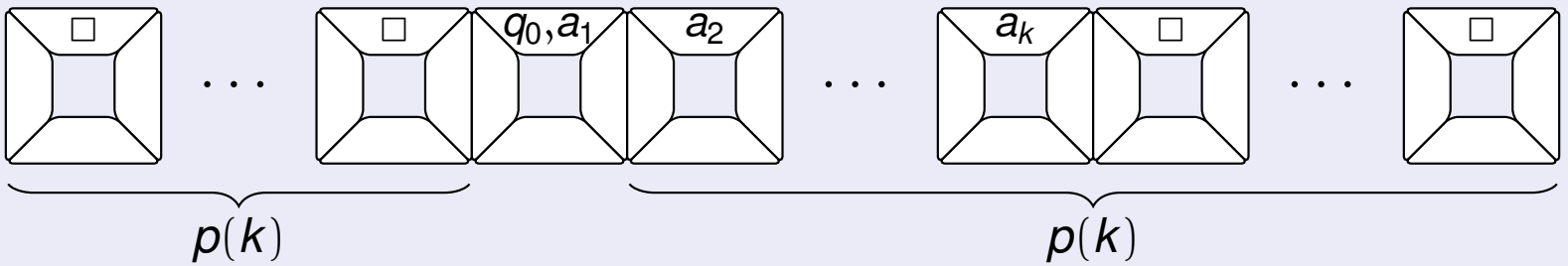
As first row we choose:



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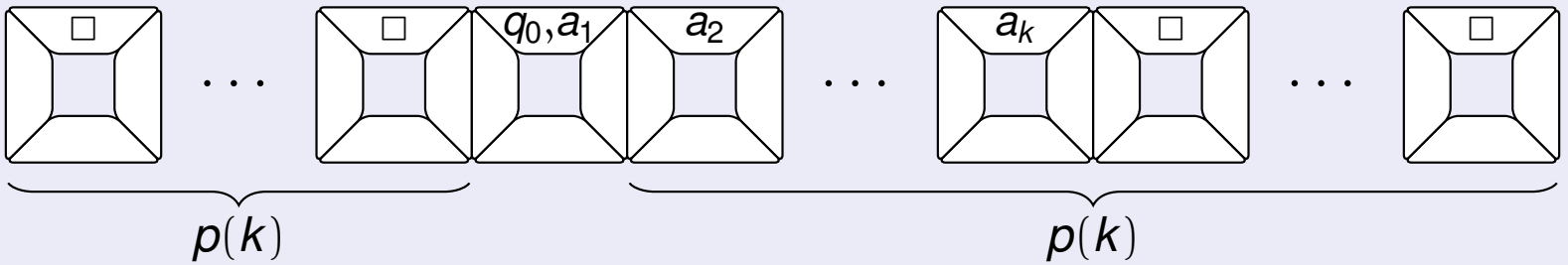
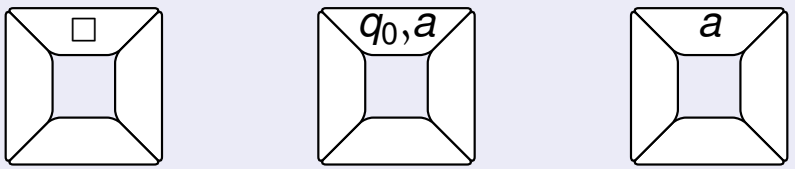


Tiles for building the first row (for every $a \in \Sigma$):



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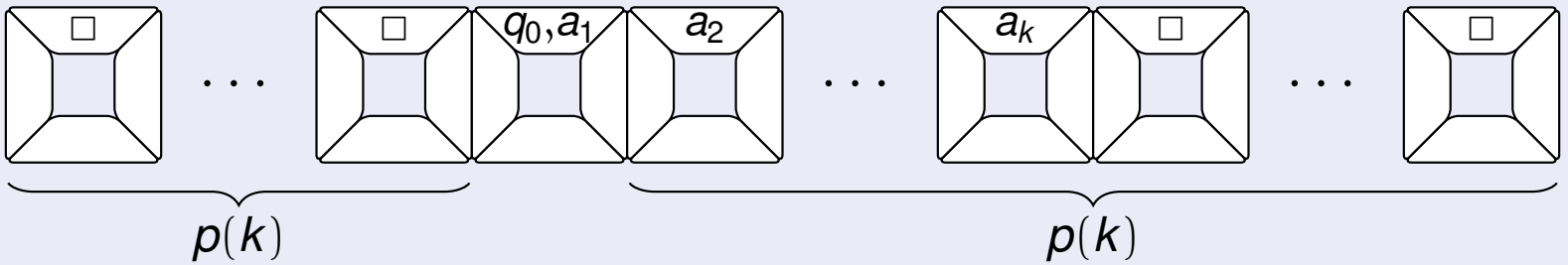

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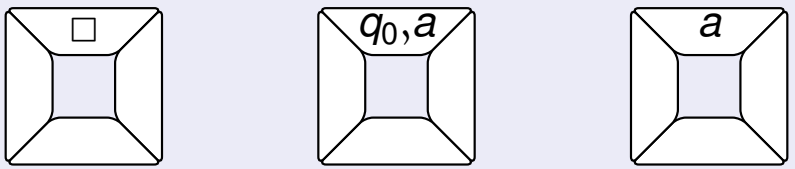

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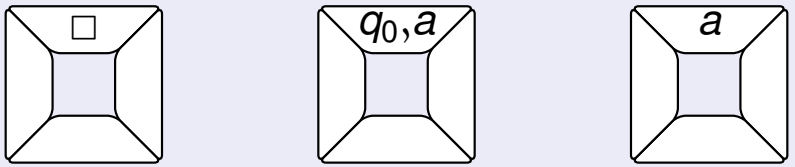

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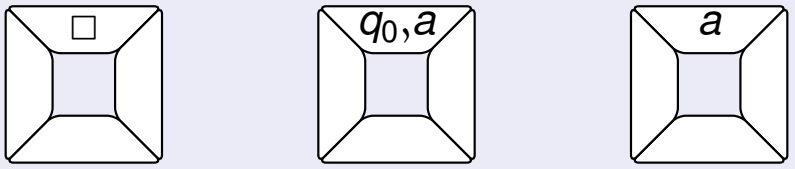
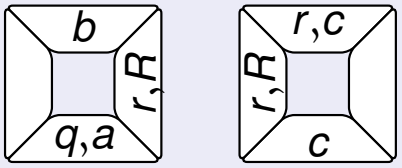
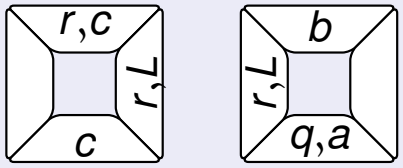


Tiles simulating the computation of M (for every $c \in \Gamma$):



for $(r, b, R) \in \delta(q, a)$



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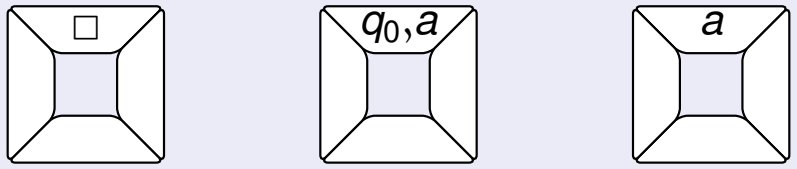

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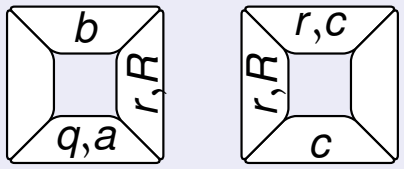

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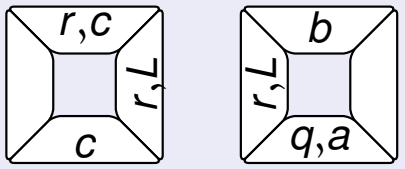
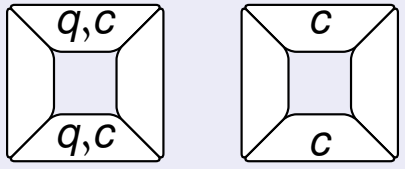


for $(r, b, L) \in \delta(q, a)$

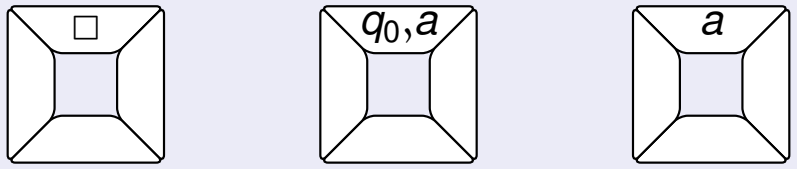
Tiles for leaving the tape unchanged (for every $q \in F$, $c \in \Gamma$):



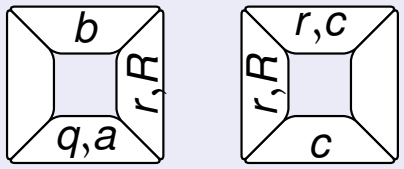
Bounded Tiling Problem is NP-complete

Proof continued. . . (the types of tiles)

Tiles for building the first row (for every $a \in \Sigma$):



Tiles simulating the computation of M (for every $c \in \Gamma$):



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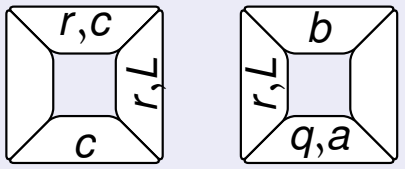

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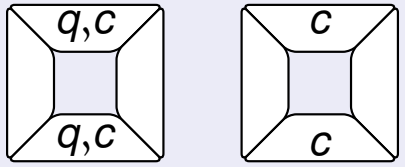

continued on the next slide. . .

Bounded Tiling Problem is NP-complete

Proof continued...

Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

$$n \times n \text{ field can be tiled} \iff x \in L(M)$$

Bounded Tiling Problem is NP-complete

Proof continued...

Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

$$n \times n \text{ field can be tiled} \iff x \in L(M)$$

Every tiling simulates a computation of M on input x.

Bounded Tiling Problem is NP-complete

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The computation takes at most $p(k)$ steps.

Bounded Tiling Problem is NP-complete

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The computation takes at most $p(k)$ steps.

So the computation fills only $p(k) < n$ rows of the tiling.

Bounded Tiling Problem is NP-complete

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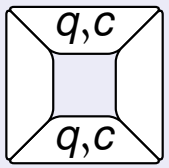
Hence, the $n \times n$ tiling can only be completed using



which exists only for $q \in F$.

Bounded Tiling Problem is NP-complete

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Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

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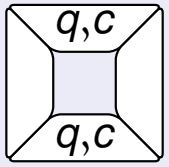

which exists only for $q \in F$.

Tiling can be finished

$\iff$ M has an accepting computation for input x.

Example

Consider the TM M with $\Sigma = \{a, b\}$, $\Gamma = \Sigma \cup \{\Box\}$, $F = \{q_1\}$ and

$$\delta(q_0, a) = \{(q_0, b, R)\} \qquad \delta(q_0, b) = \{(q_1, b, L)\}$$

Note that $L(M) = L(a^* b(a+b)^*) = L((a+b)^* b(a+b)^*)$

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For input x, M takes at most $|x|$ steps. So we take $p(k) = k$.

Example

Consider the TM M with $\Sigma = \{a, b\}$, $\Gamma = \Sigma \cup \{\square\}$, $F = \{q_1\}$ and

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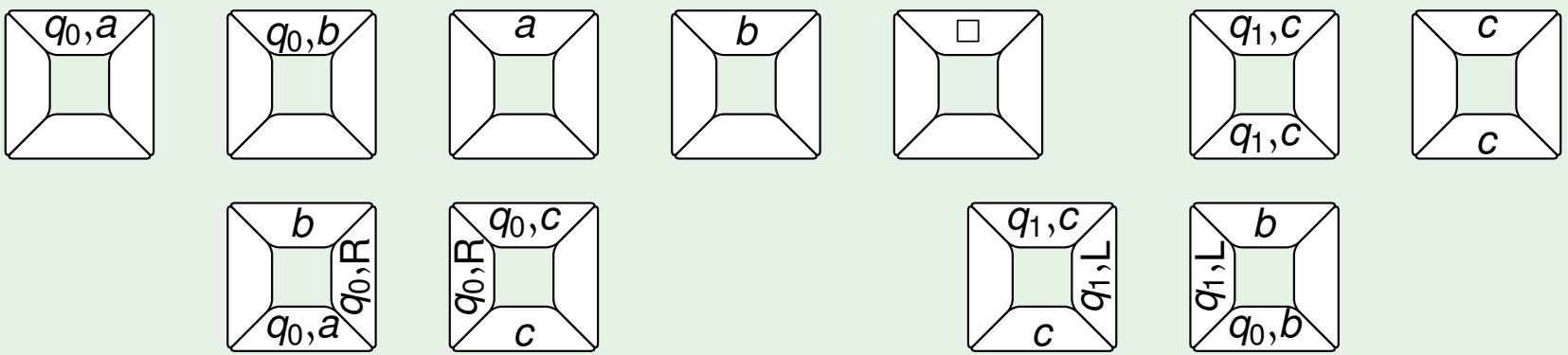
The **tile types** are:



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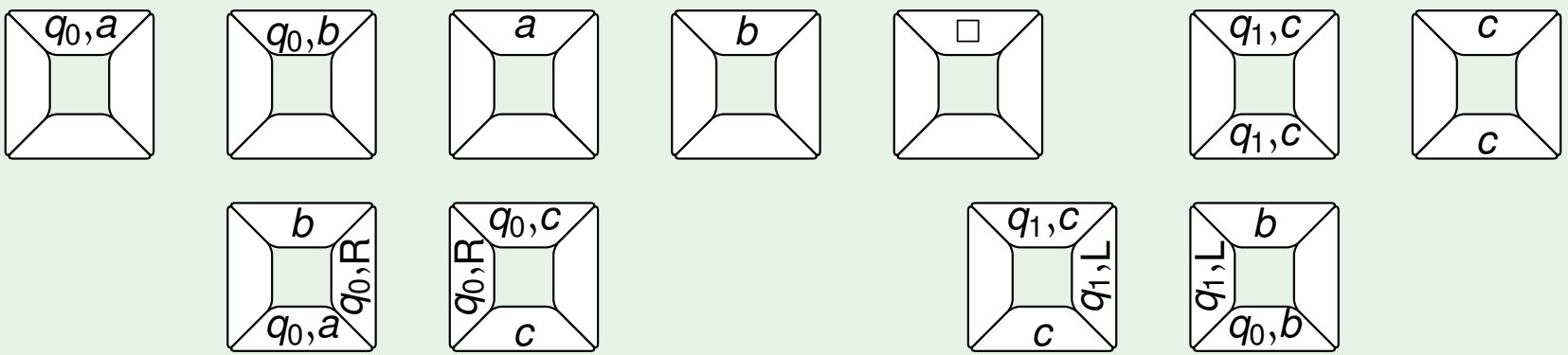

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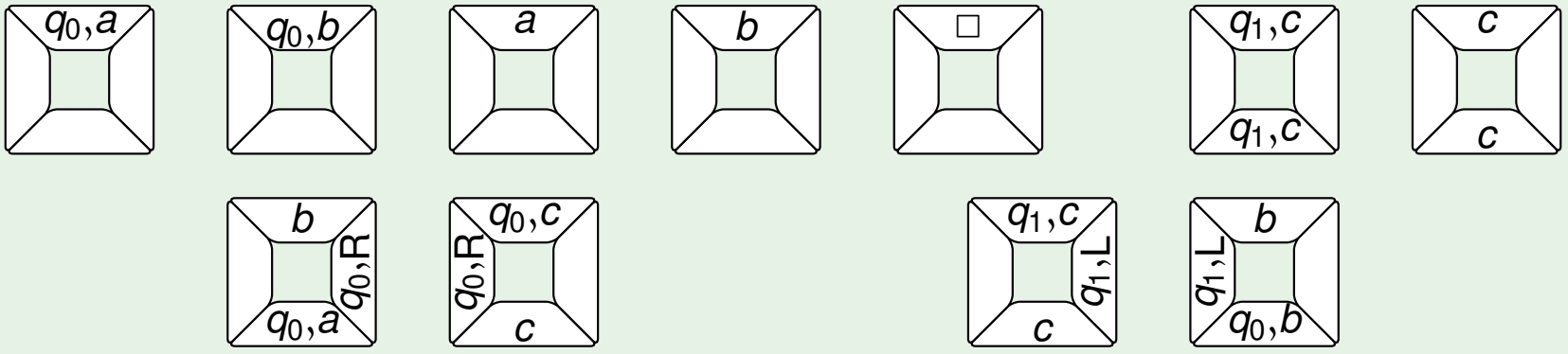
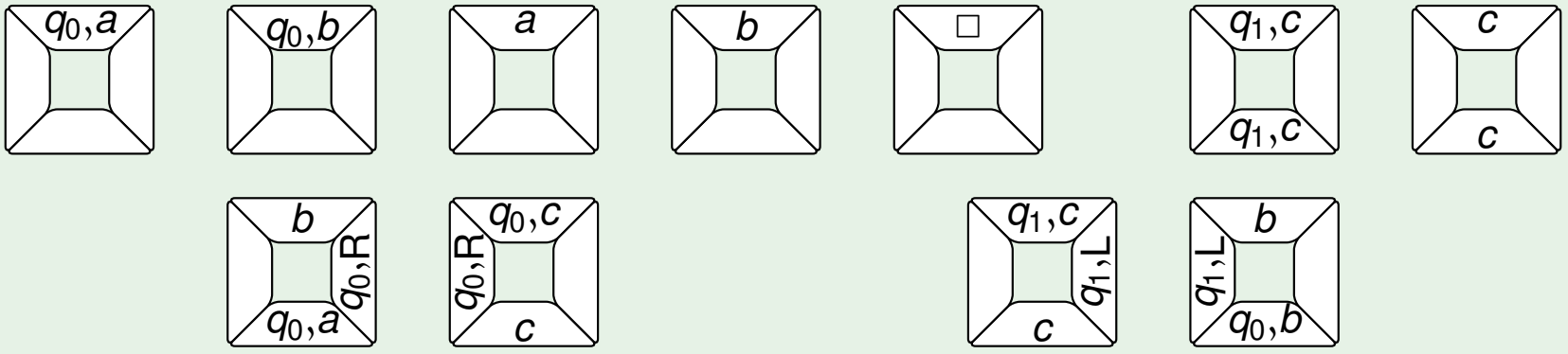
continued on the next slide. . .

Example



Consider the input word $aaa \notin L(M)$. Then $n = 2p(3) + 1 = 7$.

Example



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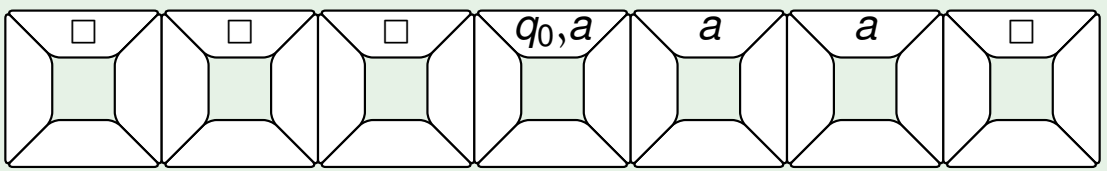

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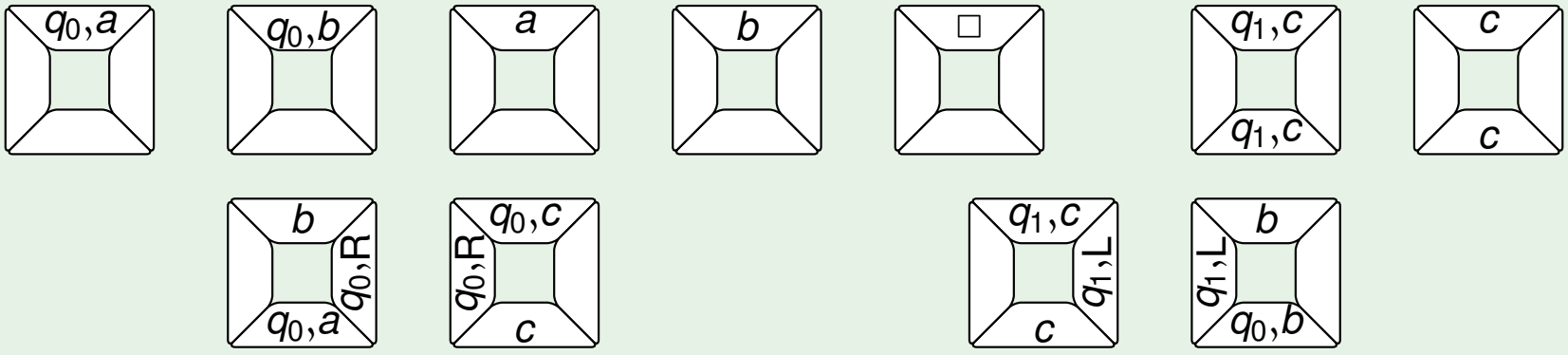

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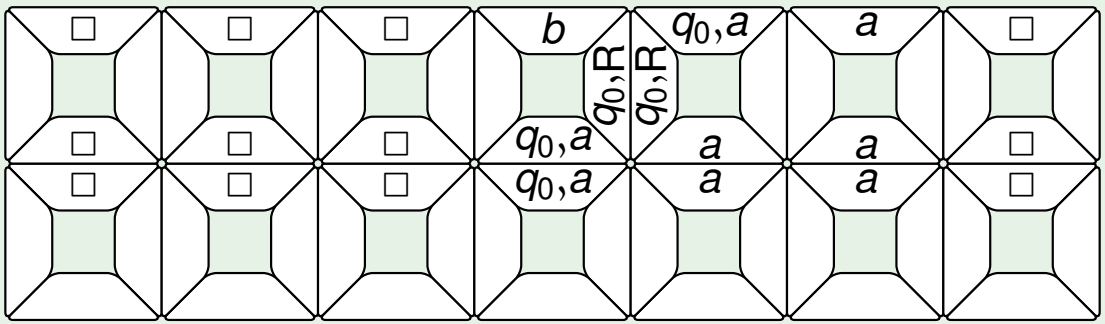

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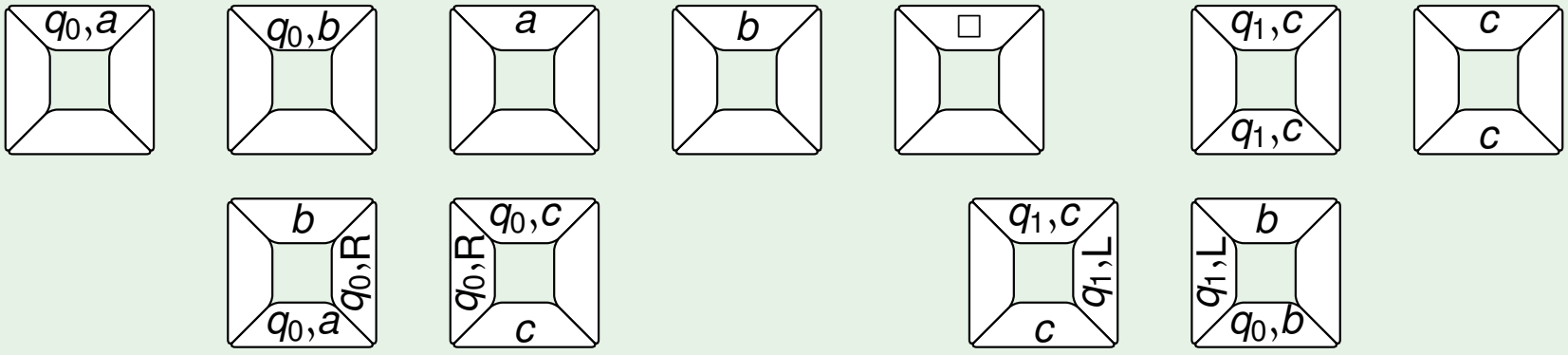

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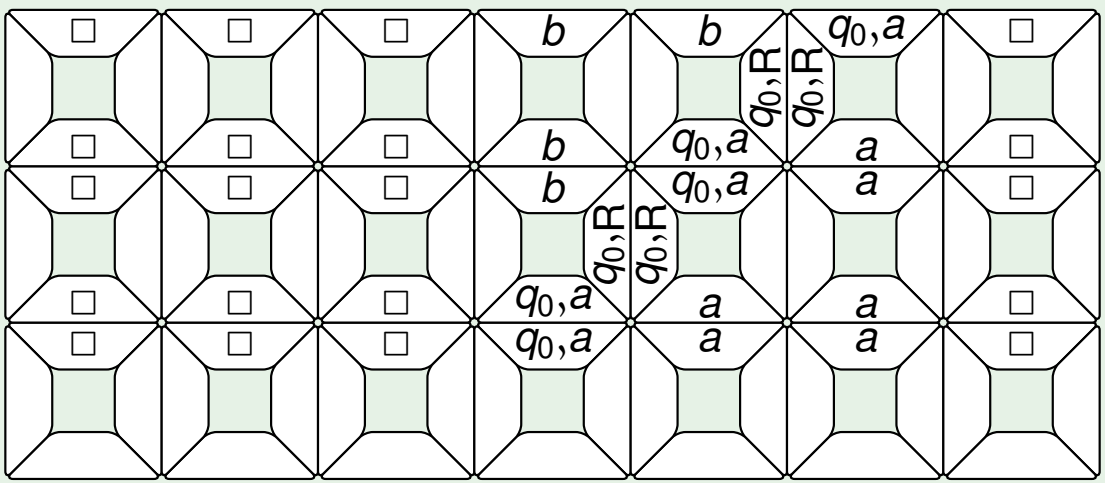

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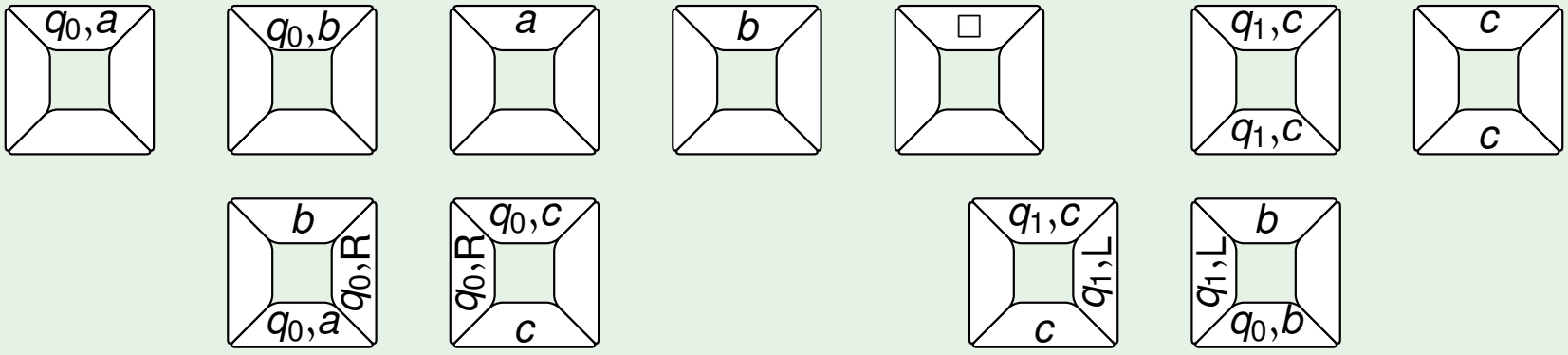

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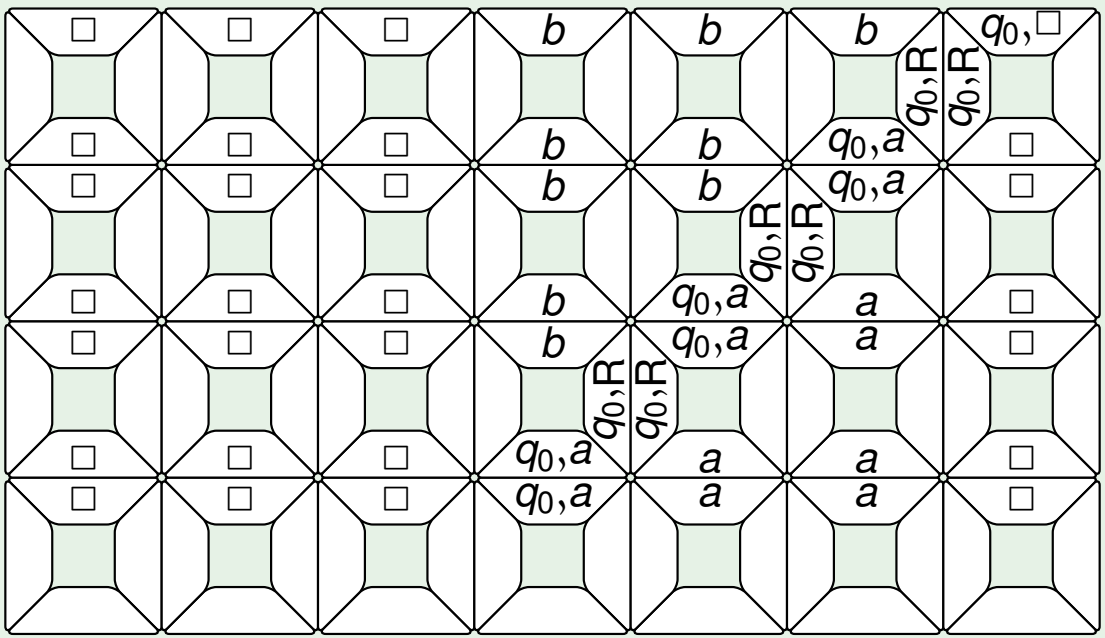

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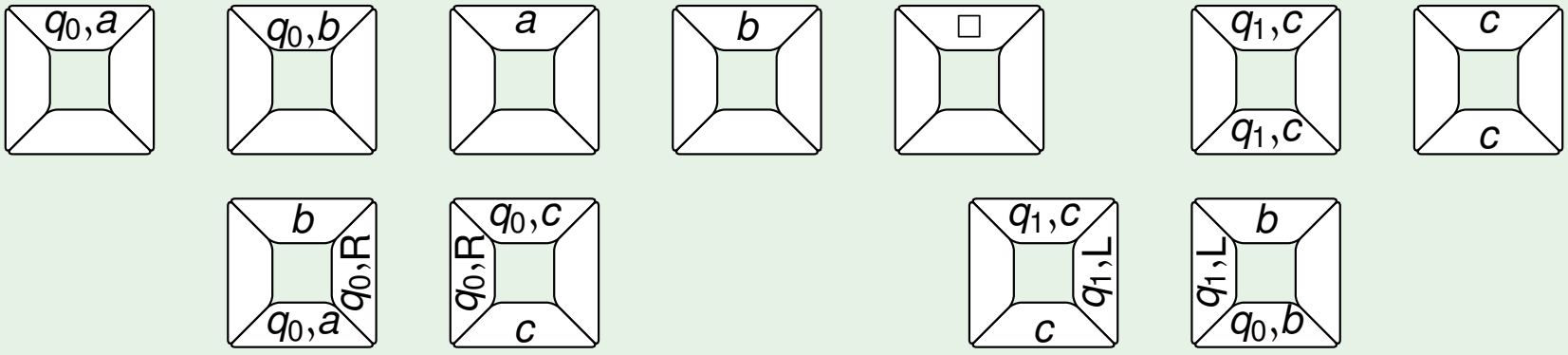

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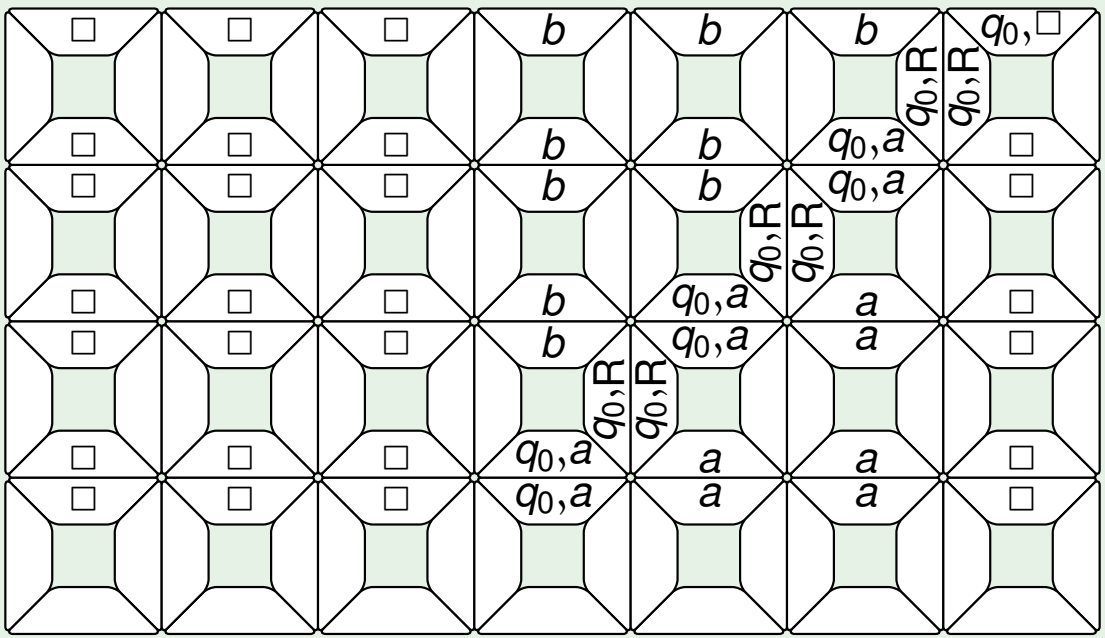

The tiling cannot be completed.

Example continued

Consider the input word $aab \in L(M)$. Then $n = 2p(3) + 1 = 7$.

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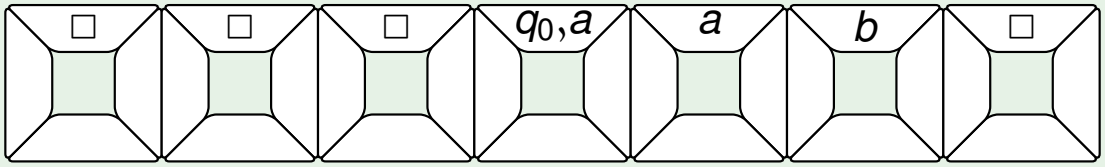

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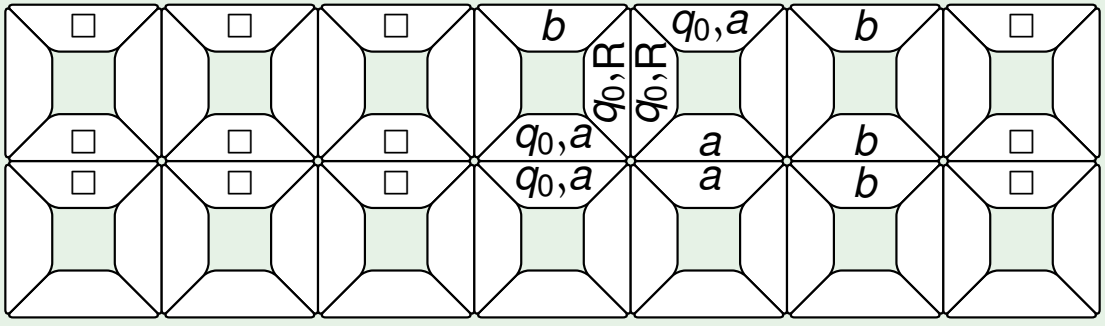

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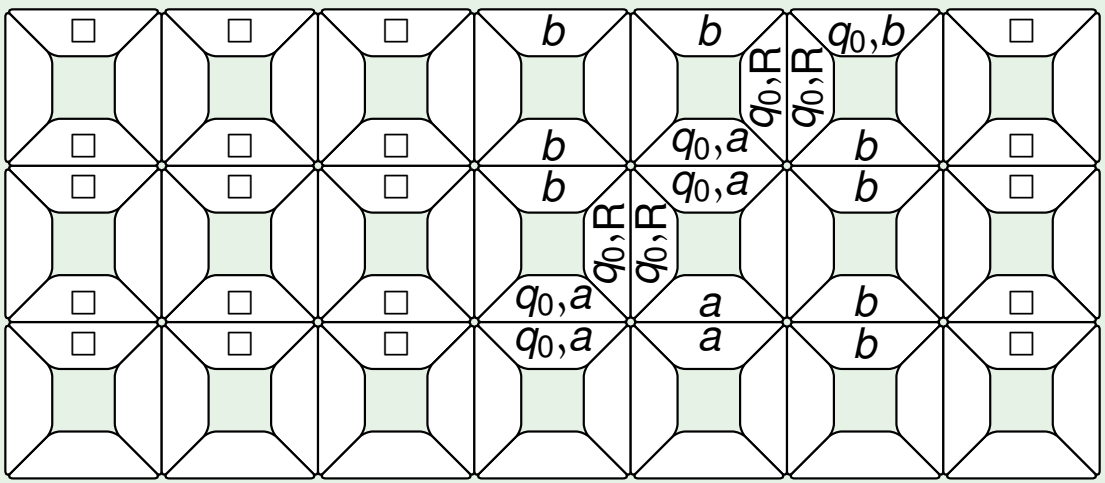

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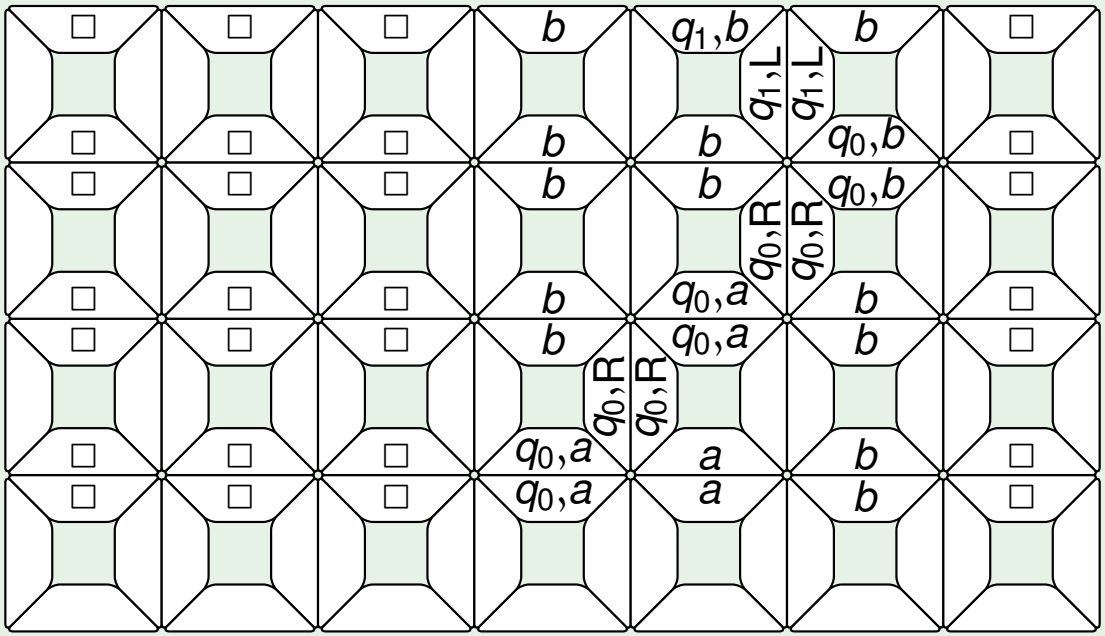

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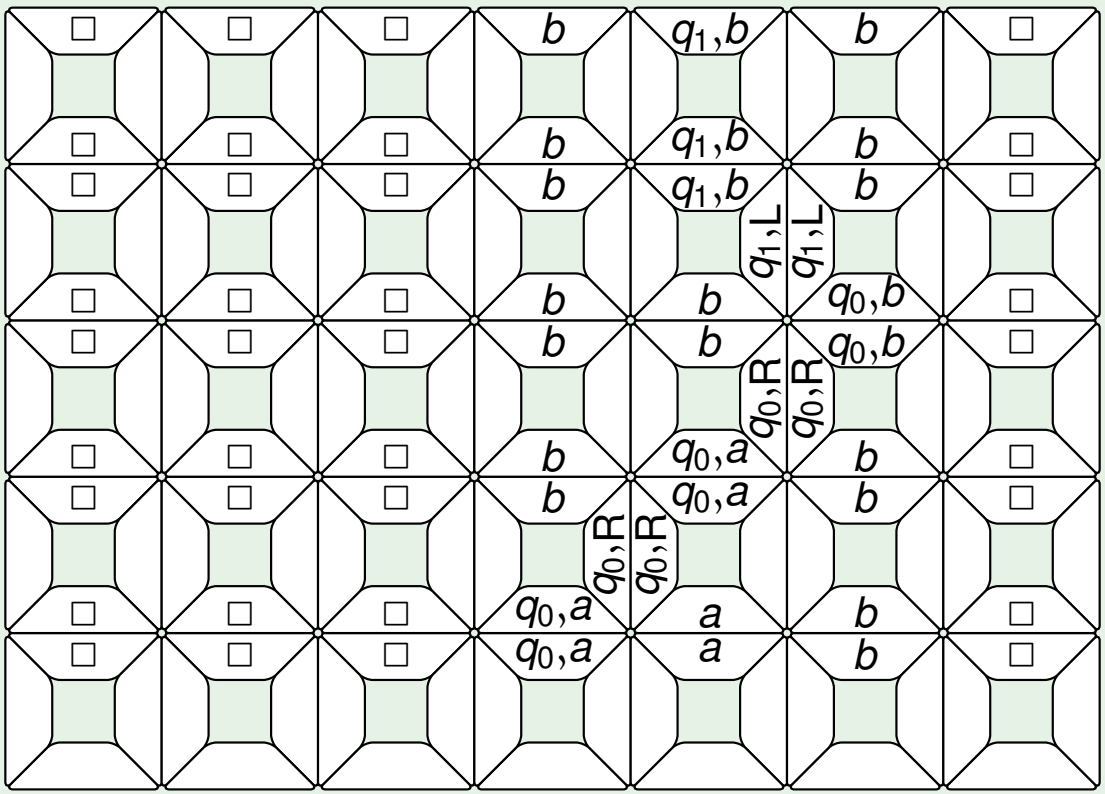

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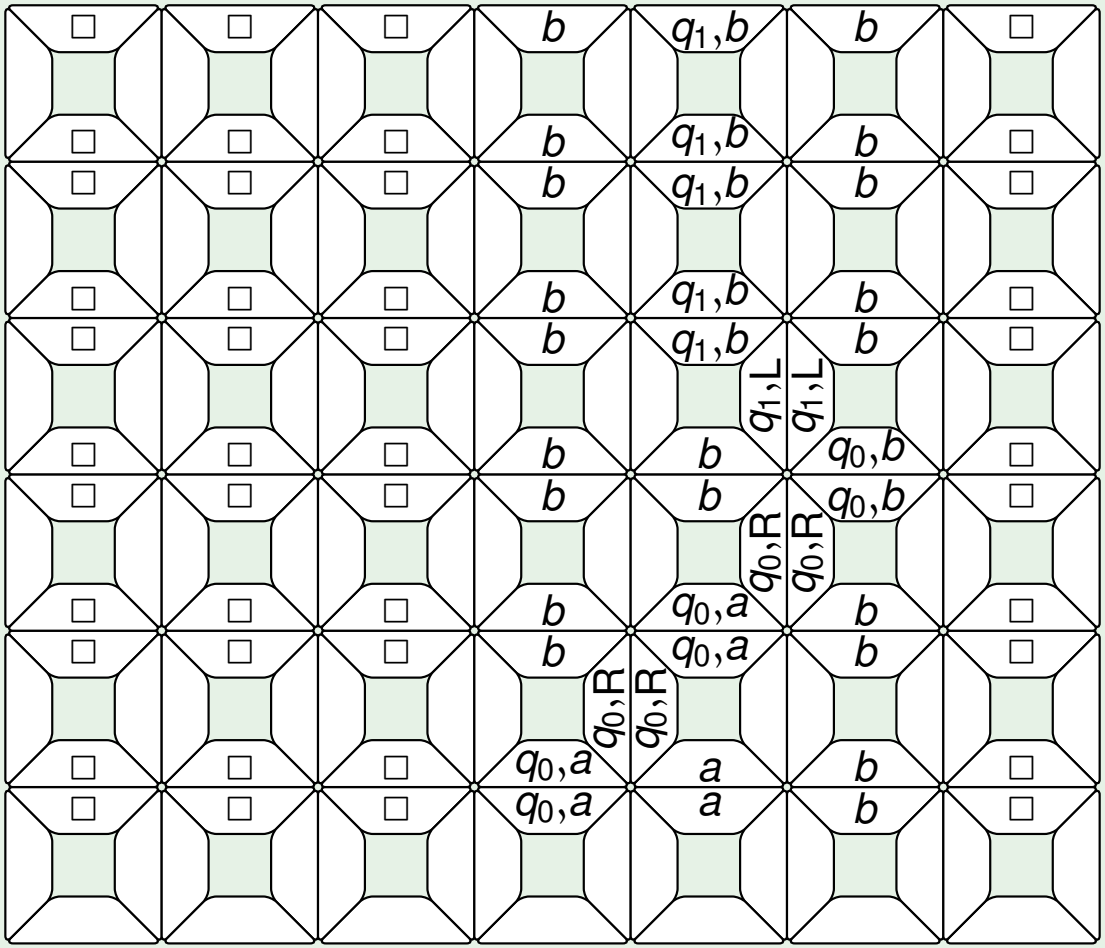

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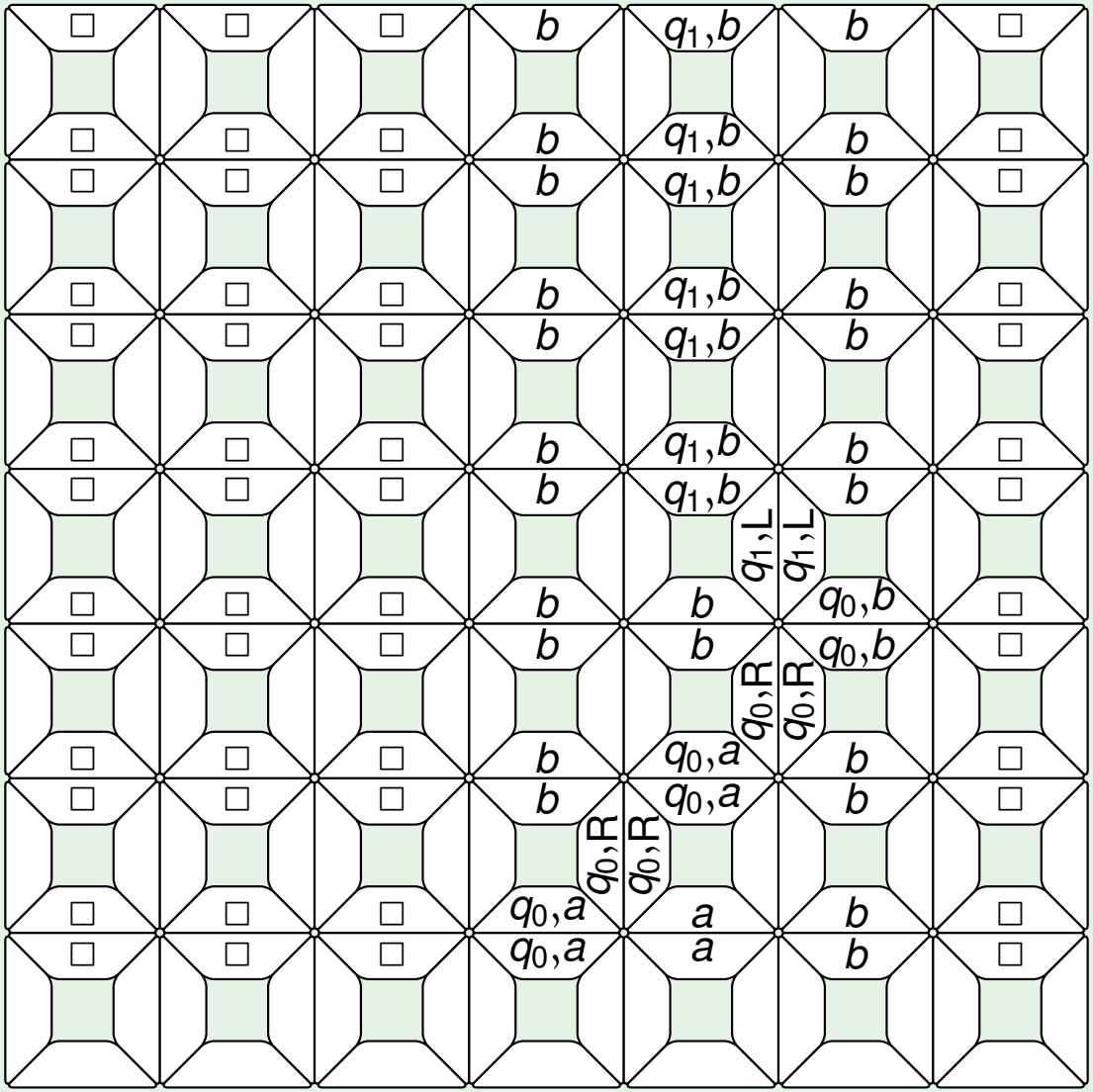

Complete tiling of the 7×7 field.

Satisfiability Problem is NP-complete

Theorem of Cook

The satisfiability problem in propositional logic is NP-complete.

Satisfiability Problem is NP-complete

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We give a polynomial-time reduction from the bounded tiling problem to the satisfiability problem.

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Given

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We create a satisfiability problem as follows.

Satisfiability Problem is NP-complete

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Intention: $x_{k\ell t} = \texttt{true} \iff$ tile of type t at position (k, ℓ).

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continued on the next slide...

Satisfiability Problem is NP-complete

Proof continued...

We define Φ to be the conjunction of the 4 formulas:

1. Fist row is $t_1 \cdots t_n$: $\bigwedge_{k=1}^{n} x_{k1t_k}$
2. At every position at most one tile type:
$$\bigwedge_{k=1}^{n} \bigwedge_{\ell=1}^{n} \bigwedge_{t \neq t'} \neg(x_{k\ell t} \wedge x_{k\ell t'})$$
3. Neighbouring tiles must match (horizontal neighbours):
$$\bigwedge_{k=1}^{n-1} \bigwedge_{\ell=1}^{n} \bigvee_{tt' \text{ matches}} (x_{k\ell t} \wedge x_{(k+1)\ell t'})$$
4. Neighbouring tiles must match (vertical neighbours):
$$\bigwedge_{k=1}^{n} \bigwedge_{\ell=1}^{n-1} \bigvee_{\substack{t' \\ t \text{ matches}}} (x_{k\ell t} \wedge x_{k(\ell+1)t'})$$

Satisfiability Problem is NP-complete

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Size of the formula is polynomial in n.

Satisfiability Problem is NP-complete

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There exists an $n \times n$ tiling with first row $t_1 \cdots t_n$
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Satisfiability Problem is NP-complete

Proof continued...

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Size of the formula is polynomial in n.

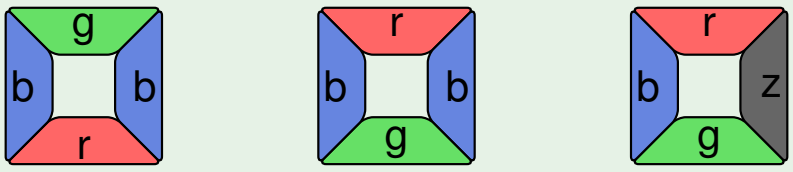
There exists an $n \times n$ tiling with first row $t_1 \cdots t_n$
$\iff$ the propositional formula Φ is satisfiable.

Thus we have a polynomial-time reduction. $\square$

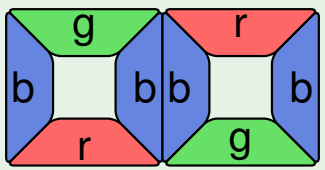
Exercise

Reduce the following instance of the bounded tiling problem to the satisfiability problem.

Types of tiles:



First row:



NP-completeness and P = NP?

Theorem

If an NP-complete language L is also in P, then P $=$ NP.

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Assume that L is NP-complete and in P.

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If an NP-complete language L is also in P, then P $=$ NP.

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Assume that L is NP-complete and in P.

Let $L' \in$ NP.

NP-completeness and P = NP?

Theorem

If an NP-complete language L is also in P, then P $=$ NP.

Proof.

Assume that L is NP-complete and in P.

Let $L' \in$ NP.

As L is NP-complete, there is a polynomial-time reduction f with

$$x \in L' \iff f(x) \in L$$

NP-completeness and P = NP?

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If an NP-complete language L is also in P, then P $=$ NP.

Proof.

Assume that L is NP-complete and in P.

Let $L' \in$ NP.

As L is NP-complete, there is a polynomial-time reduction f with

$$x \in L' \iff f(x) \in L$$

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For proving P $=$ NP it suffices to show that one NP-complete problem can be solved in deterministic polynomial time.

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Note that there are problems that are both in NP $\cap$ co-NP.

Space Complexity

Let $f, g : \mathbb{N} \to \mathbb{N}$.

A nondeterministic Turing machine M

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The function f gives an upper bound on the number of visited cells on the tape in terms of the length of the input word.

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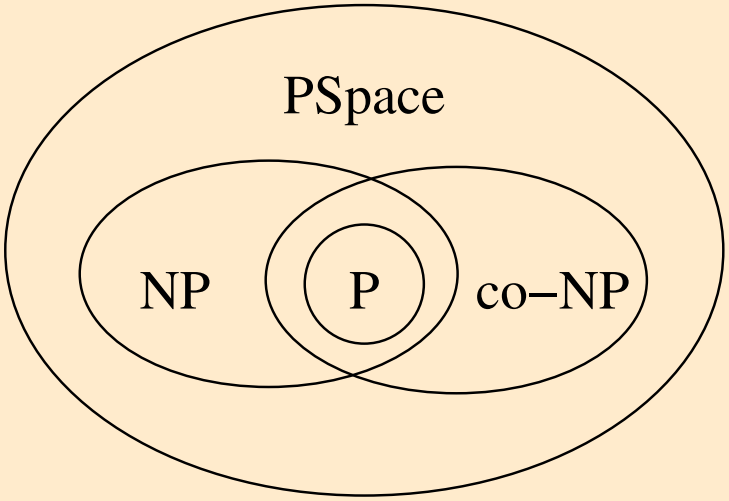
Actually, the theorem says something more general:

If L is accepted by a nondeterministic TM in $f(n)$ space, then L is accepted by a deterministic TM in $f(n)^2$ space.

PSpace-completeness



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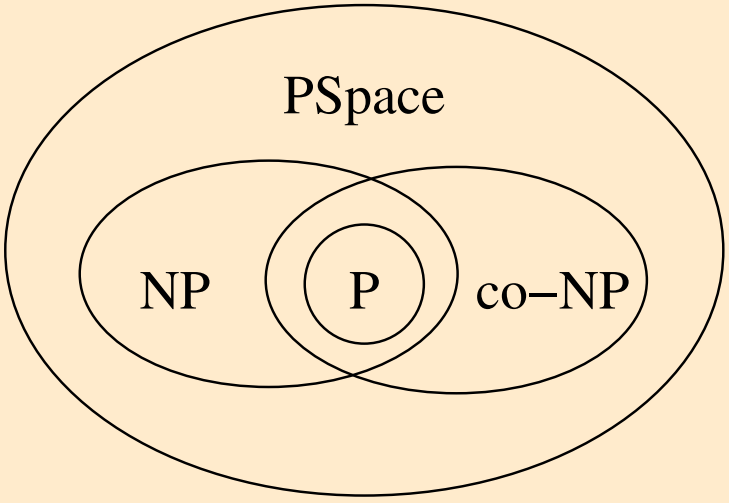

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A language $L \in$ PSpace is **PSpace-complete** if every language $L' \in$ PSpace is polynomial-time reducible to L.

$L(r) = \Sigma^*$? for regular expression r is PSpace-complete.

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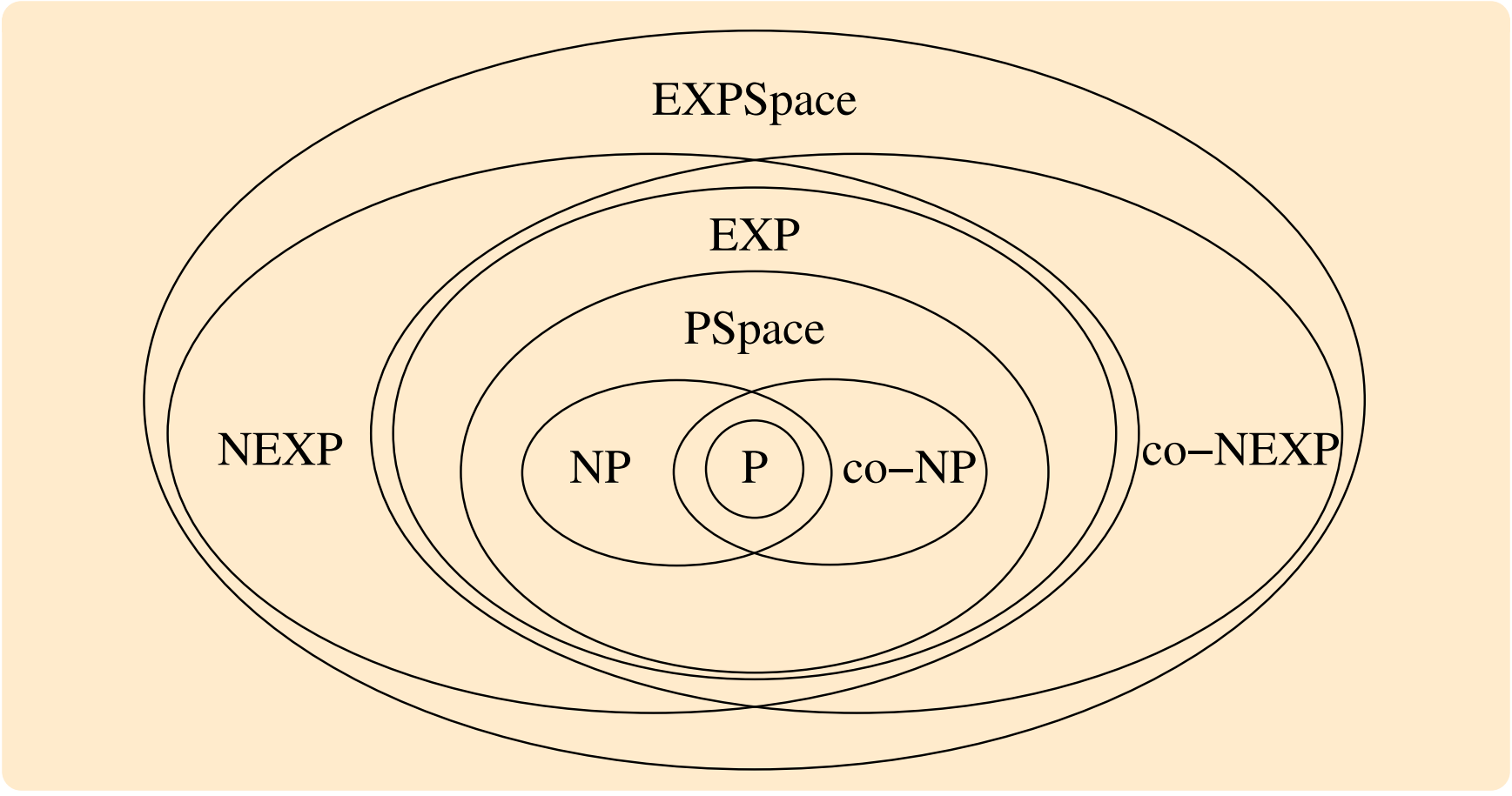
$$\text{P} \subseteq \text{NP} \subseteq \text{PSpace} \subseteq \text{EXP} \subseteq \text{NEXP}$$

It is unknown whether these inclusions are strict. We know

$$\text{P} \neq \text{EXP} \qquad \text{NP} \neq \text{NEXP}$$

PSpace $\subseteq$ EXP holds since a polynomial-space TM can at most take an exponential number of configurations.

Complexity Hierarchy



The following inclusions are known to be strict:

$$\mathrm{P} \neq \mathrm{EXP} \qquad \mathrm{NP} \neq \mathrm{NEXP} \qquad \mathrm{PSpace} \neq \mathrm{EXPSpace}$$