Automata & Complexity

Jörg Endrullis

Vrije Universiteit Amsterdam

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Decidability

A **decision problem** P is a language $P \subseteq \Sigma^*$.

The problem P is called

- **decidable** if the P is recursive, otherwise **undeciable**,
- **semidecidable** if the P is recursively enumerable.

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- algorithm that always halts
- always answers yes or no

Semidecidable problem:

- algorithm halts (eventually) it the answer is yes ($w \in P$),
- may or may not halt if the answer is no ($w \notin P$).

(Problem: one cannot know how long to wait for an answer.)

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(Semidecidable: execute M on w and wait.)

The following question not decidable and not semidecidable:

Universal halting problem

Does TM M reach a halting state on all $w \in \Sigma^*$? (Input: M.)

(The complement is also not semidecidable.)

The Halting Problem (1936)

The **halting problem** is: given

- a deterministic Turing machine M and
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Theorem

The halting problem H is undecidable.

(The language H is not recursive.)

The Halting Problem is Undecidable (Proof 1)

Proof.

Assume that there was a deterministic TM $\mathcal{H}$ that, given (M, x) decides whether M halts on x (that is, $(M, x) \in H$).

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Let M be a deterministic Turing machine and x a word.

We can decide $x \in L(M)$ as follows:

- If according to $\mathcal{H}$, M does not halt on x, then $x \notin L(M)$.
- If according to $\mathcal{H}$, M halts on x, then execute M on x to see whether $x \in L(M)$.

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Contradiction: not every recursively enumerable language is recursive. □

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p = read input;
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program *T′* (program *T* followed by):

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if result = yes
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program T' (both parts together)

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Thus T cannot exist! The halting problem is undecidable!

Theorem of Rice (1951)

A property of a class K is **trivial** if it holds for **all** or **no** $k \in K$.

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Let L_0 be a recursively enumerable language with $\neg P(L_0)$.

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For a word x, we construct a Turing machine M_x with

$$L(M_x) = \emptyset \quad \text{if } x \notin L \qquad\qquad L(M_x) = L_0 \quad \text{if } x \in L$$

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Contradiction: decidability of $P \implies L$ recursive. □

Theorem of Rice: Example

For recursively enumerable languages L, the following questions are undecidable:

1. Is $a \in L$?
2. Is L finite?

Post Correspondence Problem (1946)

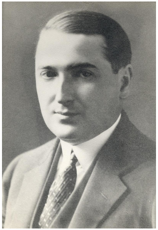
Post Correspondence Problem (PCP)

Given n pairs of words:

$$(w_1, v_1), \ \ldots, \ (w_n, v_n)$$

Are there indices $i_1, i_2 \ldots, i_k$ ($k \geq 1$) s.t.

$$w_{i_1} w_{i_2} \cdots w_{i_k} \quad = \quad v_{i_1} v_{i_2} \cdots v_{i_k} \ ?$$



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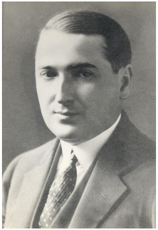

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Exercise

Find a solution for

$$(w_1, v_1) = (01, 100)$$
$$(w_2, v_2) = (1, 011)$$
$$(w_3, v_3) = (110, 1)$$

Modified Post Correspondence Problem

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$$w_1\, w_{i_1}\, w_{i_2} \cdots w_{i_k} \quad = \quad v_1\, v_{i_1}\, v_{i_2} \cdots v_{i_k} \ ?$$

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The modified PCP is undecidable.

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We define (where F and E are fresh):

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w_1 = & F & v_1 = & FS \Rightarrow & \\
w_2 = & \Rightarrow wE & v_2 = & E & \\
\vdots & x & \vdots & y & (x \to y \in P) \\
 & a & & a & (a \in T) \\
 & A & & A & (A \in V) \\
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This MPCP has a solution $\iff w \in L(G)$.

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Contradiction: If the MPCP was decidable, then $w \in L(G)$ was decidable for unrestricted grammars G! □

Example

$$S \to AA \qquad A \to aB \mid Bb \qquad BB \to aa$$

This grammar with $w = aaab$ translates to the following MPCP:

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5	A	Bb
6	BB	aa

i	w_i	v_i
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v_i :

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w_i :

1	3	7	4	10	7	8	11	5	7	8	6	9	2
F	S	$\Rightarrow$	A	A	$\Rightarrow$	a	B	A	$\Rightarrow$	a	BB	b	$\Rightarrow aaabE$

v_i :

1	3	7	4	10	7	8	11	5	7	8	6	9	2
$FS \Rightarrow$	AA	$\Rightarrow$	aB	A	$\Rightarrow$	a	B	Bb	$\Rightarrow$	a	aa	b	E

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As the MPCP is undecidable, so must be the PCP. □

Example

Consider the following instance of the MPCP:

$$w_1 = 11 \qquad w_2 = 1$$
$$v_1 = 1 \qquad v_2 = 11$$

It reduces to the following PCP problem:

$$y_0 = @\$1\$1\$ \qquad y_1 = 1\$1\$ \qquad y_2 = 1\$ \qquad y_3 = \#$$
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Example solution MPCP:

$$w_1 w_2 = 111 = v_1 v_2$$

Corresponding solution PCP:

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In general: the original MPCP instance has a solution
$\iff$ the resulting PCP instance has a solution

Undecidable Properties of Context-Free Languages

Undecidable properties of context-free languages:

- empty intersection,
- ambiguity,
- palindromes,
- equality,
- ...

Empty Intersection of Context-Free Languages

Theorem

The question $L_1 \cap L_2 = \emptyset$ **?** for context-free languages L_1, L_2 is undecidable.

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for $1 \le i \le n$. Here #, $\langle$ and $\rangle$ are fresh symbols. Then

$$L(G_1) = \{w_j \cdots w_k \# \langle k \rangle \cdots \langle j \rangle \mid 1 \le j, \ldots, k \le n\}$$
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$L(G_1) \cap L(G_2) = \emptyset \iff$ the PCP X has no solution. □

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Then G is ambiguous $\iff$ the PCP X has a solution. □

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$L(G)$ contains a palindrome $\iff$ PCP X has a solution. □

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$$S_1 \to w_i S_1 \langle i \rangle \mid w_i \# \langle i \rangle$$

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as before. Then

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Equality of Context-Free Languages

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The question $L = \Sigma^*$? (and hence $L_1 = L_2$?) for context-free languages L (L_1, L_2) is undecidable.

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Equality of Context-Free Languages (2)

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The words in $L(G_1)$ are of the form

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All these words are of the shape

$$L_S = \Sigma^* \cdot \{\#\} \cdot \{\langle 1 \rangle, \ldots, \langle n \rangle\}^+.$$

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Exercise

Give context-free grammars for L_{smaller}, L_{larger} and L_{equal}.

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There exist algorithms for these problems that always halt if the answer is yes, but may or may not halt if the answer is no.

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In 1970 **Yuri Matiyasevich** proved that this is undecidable.