Automata Theory :: Minimal DFAs

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Minimal DFAs (Hopcroft, 1971)

Goal

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$.

Construct the (unique) **minimal DFA** $\widehat{M}$ with $L(M) = L(\widehat{M})$.

(Here minimal is with respect to the number of states.)

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Step 2: Partition Q in indistinguishable states.

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Remove all states $q \in Q$ for which there is no path from q_0 to q.

Minimal DFAs (2)

States $q_1, q_2 \in Q$ are **distinguishable** if there exists $w \in \Sigma^*$ s.t.

$$q_1 \xrightarrow{w}\!\!\!\!\twoheadrightarrow q_1' \in F \qquad\qquad q_2 \xrightarrow{w}\!\!\!\!\twoheadrightarrow q_2' \not\in F \, ,$$

or vice versa.

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- If there are partitions R and S such that

$$\delta(q, a) \in S \qquad \text{and} \qquad \delta(q', a) \notin S,$$

  for some $a \in \Sigma$ and $q, q' \in R$, then we split R in

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  $$\{ q \in R \mid \delta(q, a) \in S \} \qquad \{ q \in R \mid \delta(q, a) \notin S \}$$

We keep splitting until no more split is possible.

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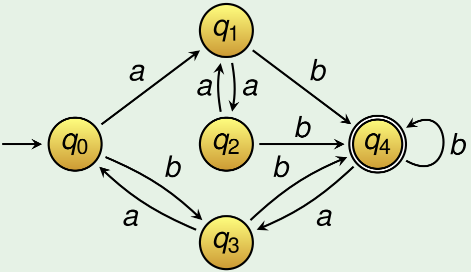
Worst-case time complexity: $O(|\Sigma| \cdot |Q|^2)$, since

- There are maximal $|Q| - 1$ splits.
- Every split costs maximal $O(|\Sigma| \cdot |Q|)$.

Exercise: DFA Minimisation



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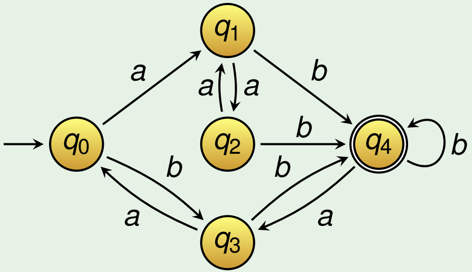

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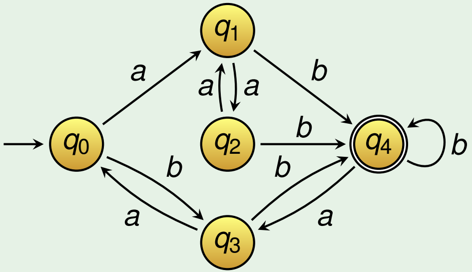

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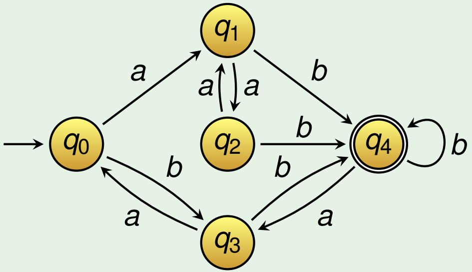

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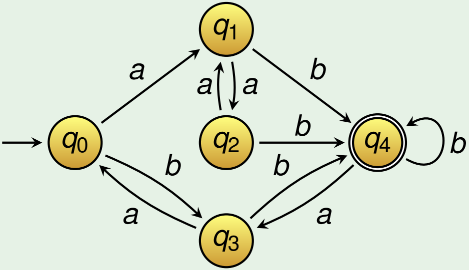

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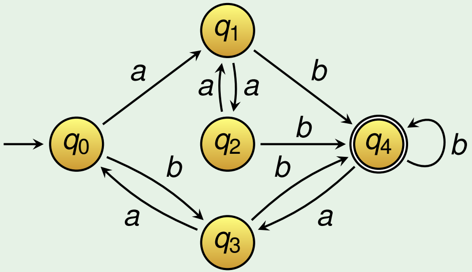

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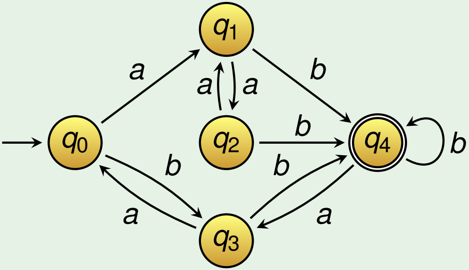

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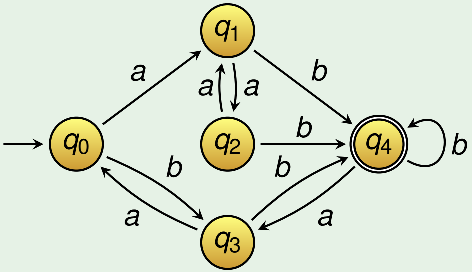

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   Nothing more to split!

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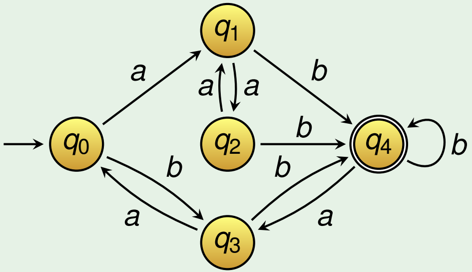

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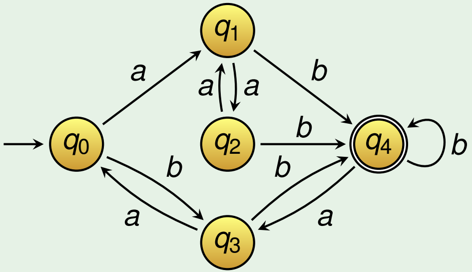
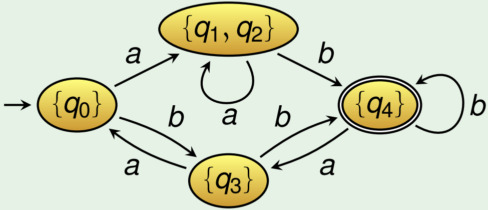


2. Final partitioning: $\{\{q_0\}, \{q_1, q_2\}, \{q_3\}, \{q_4\}\}$.
3. Reading off the minimal DFA:



Minimising of NFAs

Minimising of NFAs is very difficult.

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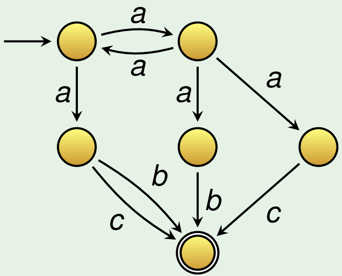

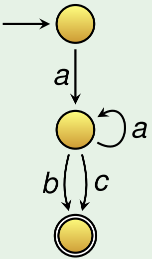

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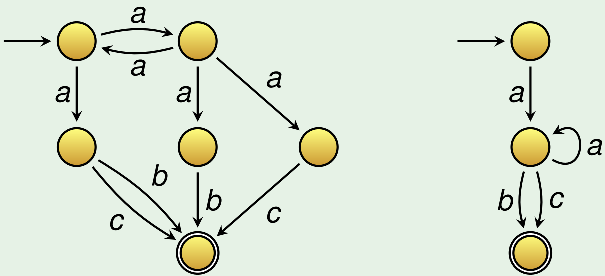

Theorem

Minimising of NFAs is PSpace-complete.

The definition of PSpace-complete follows later.

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For example the expression

```
sum = 15 + 2
```

could be converted to the sequence of tokens

token	token category
`sum`	identifier
`=`	assignment
`15`	integer literal
`+`	operator
`2`	integer literal

Allows to write parsers on the more abstract level of tokens.

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Parser/lexer generators like

- JavaCC
- LEX

generate the lexer automatically. Thereby regular expressions or grammars are converted to minimal DFAs.