Automata & Complexity

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Looking Back

Previous subjects relevant for this lecture:

- Not all languages are context-free:
  pumping lemma for context-free languages.

- Turing machines: automata with unlimited memory (tape)
  can describe languages that are not context-free.

Variations of Turing Machines

We consider the following variations of Turing machines:

- Turing machines with multiple tapes
- nondeterministic Turing machines
- universal Turing machines

Turing Machines with Multiple Tapes

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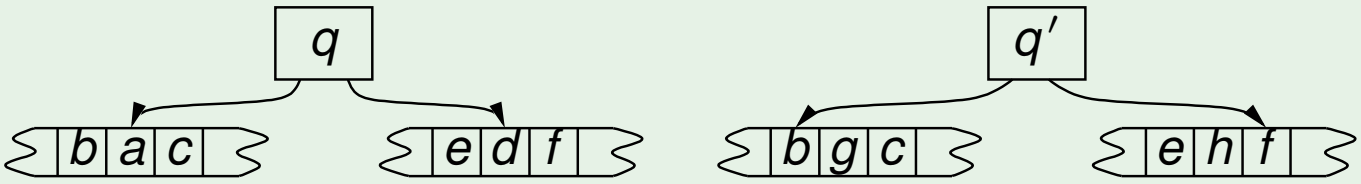

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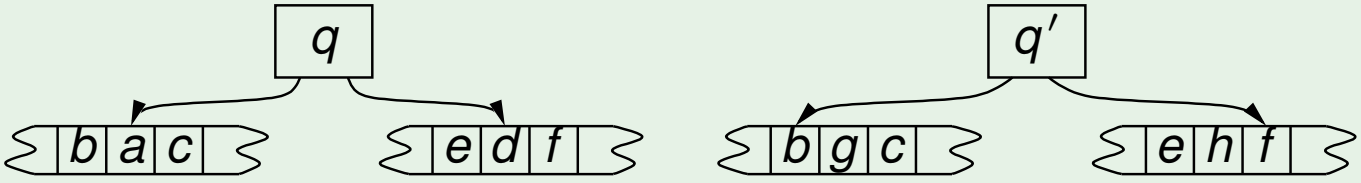
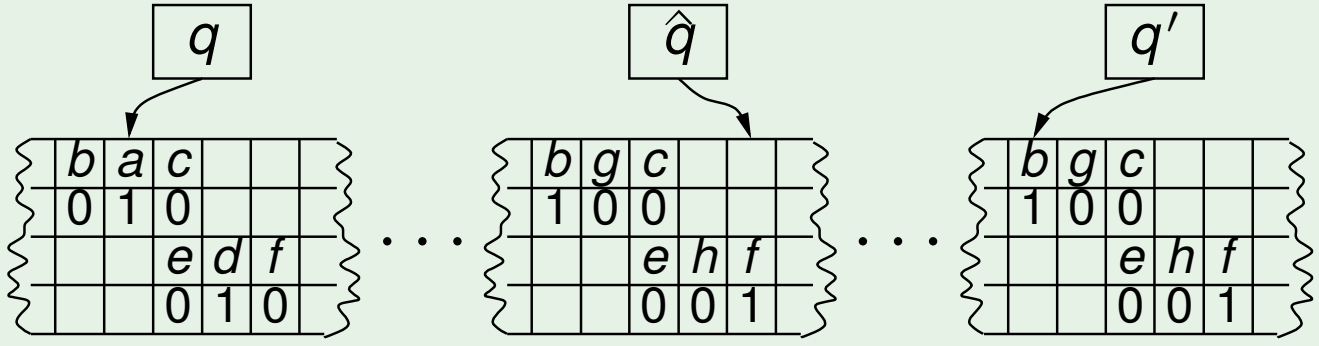


The transition translates to multiple transitions on one tape:



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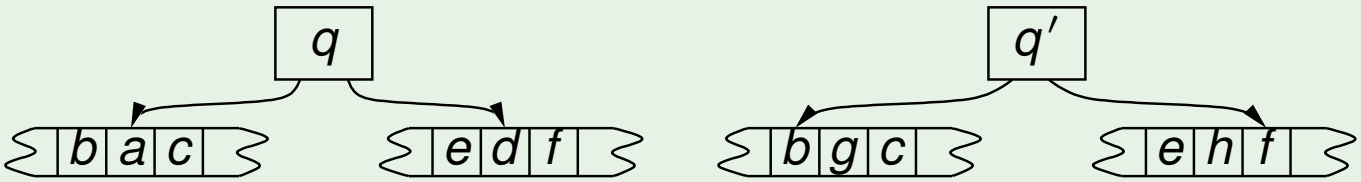

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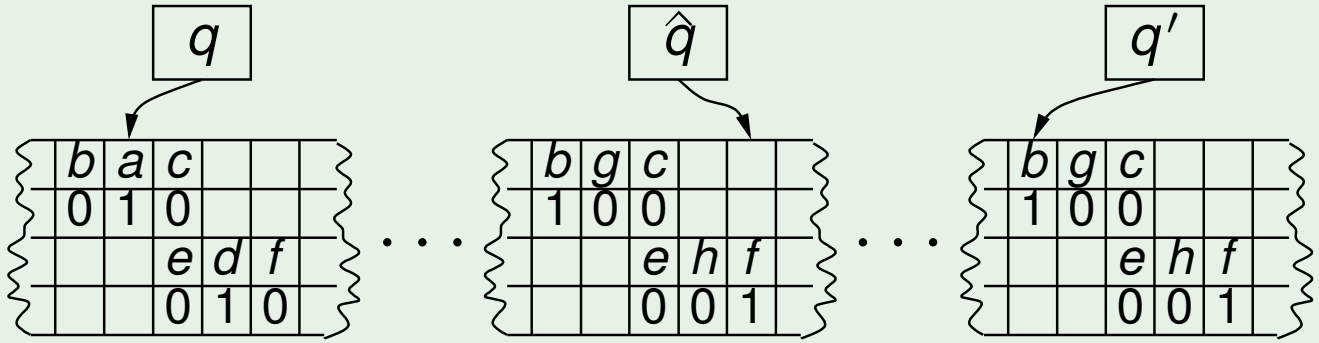

The difference in **time complexity** between a Turing machine with one tape and multiple tapes is a **polynomial** factor.

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Proof.

The deterministic TM can use

- **breadth-first search**

to simulate all executions of the nondeterministic TM in parallel.

The branches of the breadth-first search can be stored on the tape in the form **queue**. □

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The difference in **time complexity** between nondeterministic and deterministic TM is, as far as known, an **exponential** factor.

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A universal TM gets as input

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There exists a universal Turing machine.

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Theorem

A language L is generated by an unrestricted grammar
$\iff$ L is accepted by a Turing machine.

From Unrestricted Grammars to Turing Machines

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For every unrestricted G there is a Turing machine M such that

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The variables are S, T, $\square$

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Step 1: guessing the word w

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After step 1, we have derived something of the form

$$V_\square^\square \cdots V_\square^\square V_{q_0 a_1}^{a_1} V_{a_2}^{a_2} V_{a_3}^{a_3} \cdots V_{a_n}^{a_n} V_\square^\square \cdots V_\square^\square$$

where $w = a_1 a_2 \cdots a_n$.

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Next, the TM is simulated using the lower line (the subscripts).

From Turing Machines to Unrestricted Grammars

Construction continued

Step 2: simulating the TM (in the subscripts)

$$V^{\alpha}_{qc} V^{\beta}_{\gamma} \to V^{\alpha}_{d} V^{\beta}_{q'\gamma} \qquad \text{if } \delta(q, c) = (q', d, R)$$

$$V^{\beta}_{\gamma} V^{\alpha}_{qc} \to V^{\beta}_{q'\gamma} V^{\alpha}_{d} \qquad \text{if } \delta(q, c) = (q', d, L)$$

for every $\alpha, \beta \in \Sigma \cup \{\square\}$ and $\gamma \in \Gamma$.

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Step 3: If TM reaches accepting state, then generate w.
(From the superscripts left unchanged in step 2.)

$$V_{q\gamma}^{\alpha} \to \alpha \qquad \text{for every } q \in F$$

$$\beta V_{\gamma}^{\alpha} \to \beta\alpha$$

$$V_{\gamma}^{\alpha} \beta \to \alpha\beta$$

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for every $\alpha, \beta \in \Sigma \cup \{\square\}$ and $\gamma \in \Gamma$.

Then $L(G) = L(M)$.

Example

Consider the TM with $\Sigma = \{a, b, c\}$, $\Gamma = \Sigma \cup \{\Box\}$, $F = \{q_2\}$ and

$$\delta(q_0, a) = (q_0, c, R) \qquad \delta(q_0, c) = (q_1, b, L)$$
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This TM accepts the language $L((a + b)^* bc(a + b + c)^*)$.

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$$V_{q_0 b}^{\alpha} V_{\gamma}^{\beta} \to V_b^{\alpha} V_{q_0 \gamma}^{\beta} \qquad \beta V_{\gamma}^{\alpha} \to \beta \alpha$$
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$$V_\gamma^\beta V_{q_0 c}^\alpha \to V_{q_1 \gamma}^\beta V_b^\alpha \qquad V_\gamma^\alpha \beta \to \alpha\beta$$
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with $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$. Exercise: derive abc.

Context-Sensitive Grammars

A grammar is **context-sensitive** if for every rule $x \to y$ it holds:

$$|x| \leq |y| \quad \text{(and } x \neq \lambda\text{)}$$

Note: the words cannot get shorter during derivation.

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For every context-sensitive grammar G_1
there exists a grammar G_2 with rules of the form

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such that $L(G_1) = L(G_2)$.

(Compare with the shape of rules in a context-free grammar.)

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A language L is **context-sensitive** if there exists a context-sensitive grammar G with $L(G) = L \setminus \{\lambda\}$.

Example

The language

$$\{ a^n b^n c^n \mid n \geq 1 \}$$

is generated by the context-sensitive grammar:

$$S \rightarrow aAbc \mid abc$$
$$A \rightarrow aAB \mid aB$$
$$Bb \rightarrow bB$$
$$Bc \rightarrow bcc$$

Example derivation:

$$\begin{aligned} S &\Rightarrow aAbc \Rightarrow aaABbc \Rightarrow aaAbBc \\ &\Rightarrow aaAbbcc \Rightarrow aaaBbbcc \Rightarrow aaabBbcc \\ &\Rightarrow aaabbBcc \Rightarrow aaabbbccc \end{aligned}$$

Linear Bounded Automata

A **linear bounded automaton**, short **LBA**,
is a nondeterministic TM $(Q, \Sigma, \Gamma, \delta, q_0, F)$.

Note that there is no $\square$!

Instead, we have symbols $[$ and $]$, and

- $[$ and $]$ are placed around the input word
- for every $q \in Q$, $\delta(q, [\,)$ is of the form $(q', [\,, R)$
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The head can only move within the bounds of the input word!

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The language $L(M)$ accepted by LBA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is

$$\{\, w \in \Sigma^+ \mid q_0[w] \vdash^+ [uqv] \text{ for some } q \in F, u, v \in \Gamma^* \,\}$$

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Proof.

A derivation of $w \in L(G)$ contains only words of length $\leq |w|$.

A nondeterministic Turing machine can simulate (guess) this derivation without leaving the bounds of w. $\square$

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(In step 1, we derive from S a word $V_{q_0[a_1}^{a_1} V_{a_2}^{a_2} \cdots V_{a_{n-1}}^{a_{n-1}} V_{a_n]}^{a_n}$.)

Basic Properties of Context-Sensitive Languages

Theorem

If L_1 and L_2 are context-sensitive, then so are

$$L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1^R \qquad L_1 L_2 \qquad L_1^* \qquad \overline{L_1} \qquad L_1 \setminus L_2$$

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- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$

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- $L_1 \cap L_2$: using linear bounded automata
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$
- $\overline{L_1}$: proven by Immerman and Szelepcsényi (1987) $\square$

Basic Properties of Context-Sensitive Languages

Theorem

If L_1 and L_2 are context-sensitive, then so are

$$L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1^R \qquad L_1 L_2 \qquad L_1^* \qquad \overline{L_1} \qquad L_1 \setminus L_2$$

Proof.

- $L_1 \cup L_2$, L_1^R, $L_1 L_2$, L_1^*: using grammars (or automata)
- $L_1 \cap L_2$: using linear bounded automata
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$
- $\overline{L_1}$: proven by Immerman and Szelepcsényi (1987) □

It is unknown whether **deterministic** LBA's are equally expressive as **nondeterministic** LBA's.

Recursively Enumerable Languages

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Then $w_1, w_2, \ldots$ is also called a **recursive enumeration** of L.

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- A Turing machine can be represented as a word.
- A parser can check whether a word represents a TM. (If so, accept.)

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Thus: there is a recursive enumeration $M_1, M_2, \ldots$ of all TMs.

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Theorem

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Contradiction. Hence $\overline{L}$ is not recursively enumerable. $\square$

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Proof.

See last slide. □

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Then $L(N) = \overline{L(M)}$, thus $L(M)$ is recursive. $\square$

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Then $L(M) = L(M_1) = L$, and M halts for every input. $\square$

Properties of Recursive Languages

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Proof.

- $L_1 \cup L_2$, $\overline{L}$, $L_1 L_2$, L^*: use (non)deterministic TMs
- $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$

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Then M accepts $L(G)$ and always reaches a halting state. □

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Not every recursive language is context-sensitive.

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Proof.

$\Sigma = \{0, 1\}$. There exists an **injective, computable function**

$$h : \{\, G \mid G \text{ context-sensitive} \,\} \to \{\, 0, 1 \,\}^*$$

such that the **image** of h is **recursive**. For example:

$$h(0) = 010 \qquad h(\to) = 01110 \qquad h(A_i) = 01^{i+4}0$$
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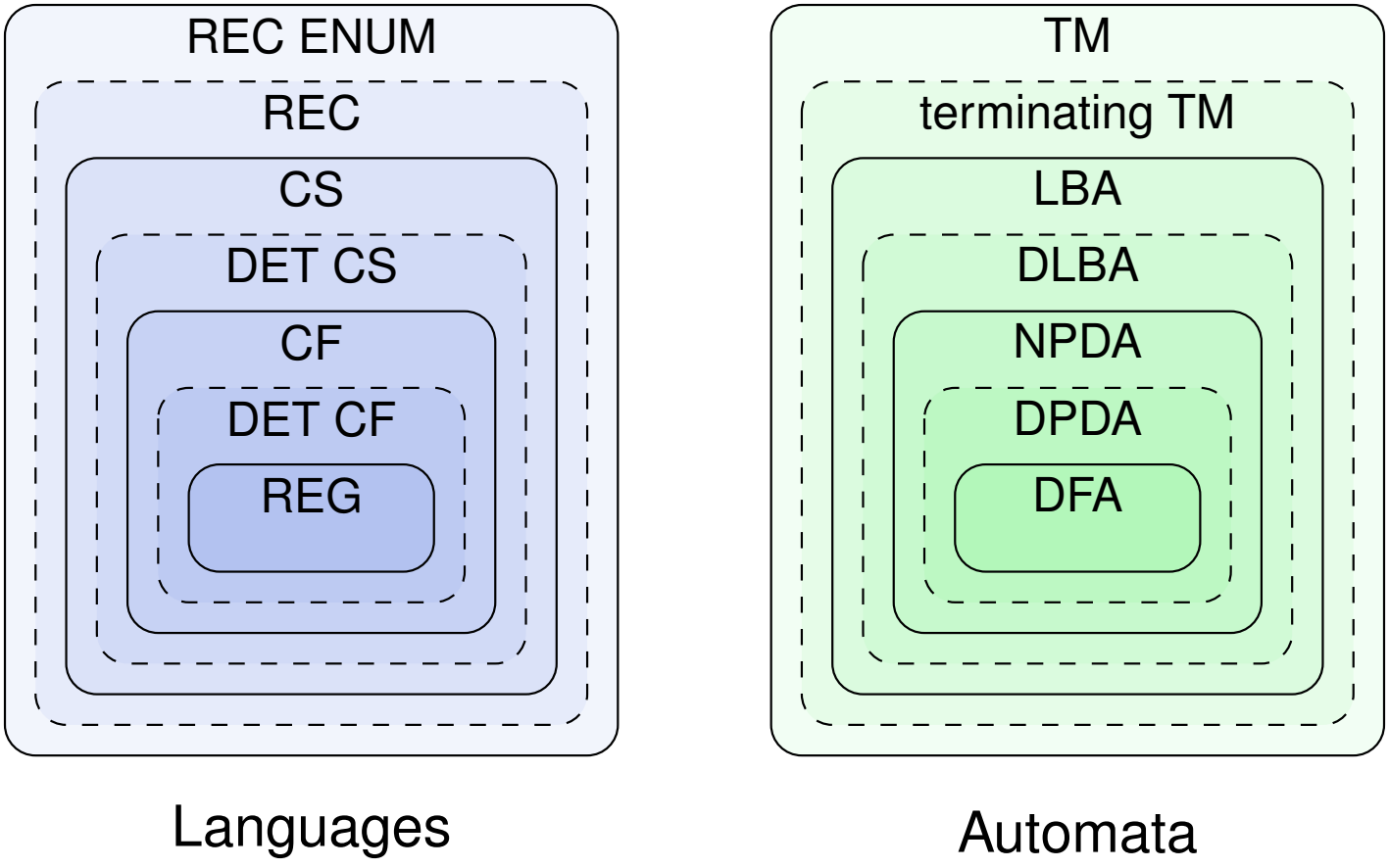
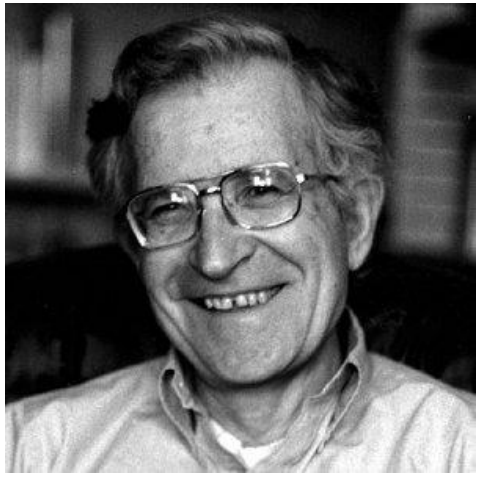
Contradiction! □

Chomsky Hierarchy



Noam Chomsky



Noam Chomsky (born 1928) is a world famous

- language expert,
- philosopher, and
- political activist.

Turing Machines are Countable, Languages not

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There are countably many TMs over an input alphabet Σ.

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There are uncountable many languages over Σ.

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Thus L is **not** part of the above enumeration. Contradiction. $\square$

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Conclusion: not all languages are recursively enumerable.

Looking Back

- Extensions of TMs do not add expressivity
- Unrestricted grammars are equivalent to TMs
- Recursively enumerable languages are described by TMs
- Recursive languages: TM always reaches a halting state
- Linear bounded automaton (LBA) is a TM without $\square$
- Context-sensitive grammars are equivalent to LBA's
- Chomsky hierarchy

Looking Forward

Read:

- Linz 10.2–10.3, 10.5, 11.1–11.4

Do the following exercises:

- Linz 11.1: 3, 6, 7, 8, 10
- Linz 11.2: 6, 7
- Linz 11.3: 1b,d, 3
- Linz 10.5: 4c,d

Following lecture:

- Decidability and undecidability
- Halting problem
- Theorem of Rice