Automata Theory :: Regular Languages

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Alternative Descriptions of Regular Languages

Recall that:

The following statements are equivalent:

- The language L is regular.
- There is a DFA M with $L(M) = L$.
- There is an NFA M with $L(M) = L$.
- There is a right linear grammar G with $L(G) = L$.
- There is a left linear grammar G with $L(G) = L$.
- There is a regular expression r with $L(r) = L$.

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- $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ is regular.
- $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$ is regular.
- For L^R, take an NFA that accepts L. Reverse all arrows. Swap final and starting states. The result accepts L^R.

□

Decidable Properties of Regular Languages

Decidability of Emptyness

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Theorem

It is decidable whether a regular language L is empty.

Proof.

- Construct a DFA (or NFA) M with $L(M) = L$.
- Check if M has a path from starting state to a final state.
- If **yes**, then $L \neq \varnothing$. If **no**, then $L = \varnothing$. □

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Solution

On-the-fly generation of DFA prevents state-space explosion. We only generate those states visited when reading *u*.

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Finally, emptyness is decidable. $\square$

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Both problems on the right are decidable. □