Automata & Complexity

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Looking Back

Previous subjects relevant for this lecture:

- Context-free languages via context-free grammars
- Context-free languages via pushdown automata

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Question

Are all languages context-free?

Pumping Lemma for Context-Free Languages (1961)

Theorem

Let L be a context-free language.

There exists $m > 0$ such that for every word $w \in L$ with $|w| \geq m$:

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with $|vxy| \leq m$ and $|vy| \geq 1$, and $uv^ixy^iz \in L$ for every $i \geq 0$.

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Let G be a context-free grammar with $L(G) = L \setminus \{\lambda\}$

- with k variables, and
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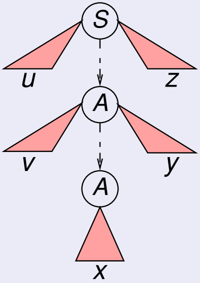

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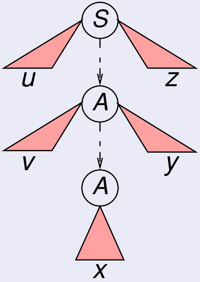

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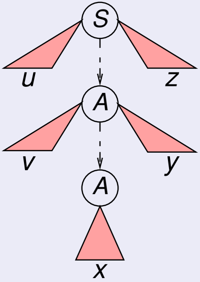

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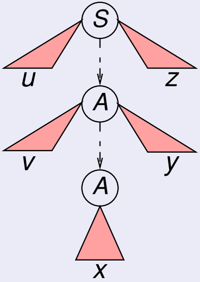

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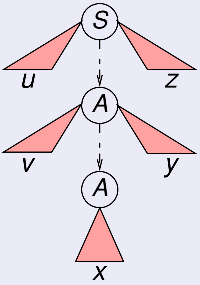

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- $|vxy| \leq m$ as the subtree generating vxy has depth $\leq k+1$,
- $|vy| \geq 1$, since there are no λ and unit productions. □

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If we can always win, then L does not fulfil the pumping lemma!

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Contradiction, thus L is not context-free.

Exercises

Show that the pumping lemma holds for:

- $\{a^n b^n \mid n \geq 0\}$
- $\{ww^R \mid w \in \{a, b\}^*\}$
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Argue that the pumping lemma does **not** hold for:

- $\{ww \mid w \in \{a, b\}^*\}$
- $\{a^{2^k} \mid k \geq 0\}$

Basic Properties of Context-Free Languages

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(for context free languages L_1 and L_2)

Namely, we have:

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}} \qquad \overline{L_1} = \Sigma^* \setminus L_1$$

Basic Properties of Context-Free Languages

Theorem

If L_1 is context-free and L_2 regular, then $L_1 \cap L_2$ is context-free.

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Then $L(\widehat{M}) = L(M) \cap L(N)$.

Question

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Why does the construction not work for two NPDA's?
(instead of an NPDA and a DFA)

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$L_2 \backslash L_1$ is **not** always context-free. Namely

$$\overline{L_1} = \Sigma^* \backslash L_1$$

Applications

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Contradiction. Thus L is not context-free.

Basic Questions about Context-Free Grammars

Given context-free grammar G and H.

Which of the following questions are **decidable**?

1. Given $w \in \Sigma^*$, do we have $w \in L(G)$?
2. Is $L(G)$ empty ?
3. Does $L(G) = \Sigma^*$ hold ?
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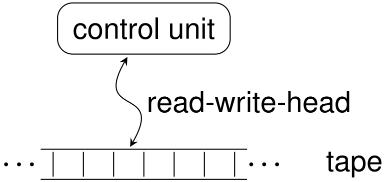
Surprisingly all other questions are undecidable.

Turing Machines

Turing machines can read and write the input word.

Input is written on a tape on which a read-write-head works.



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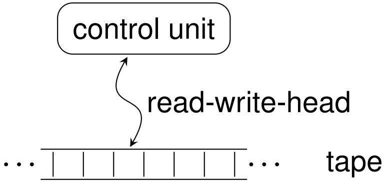

In each step:

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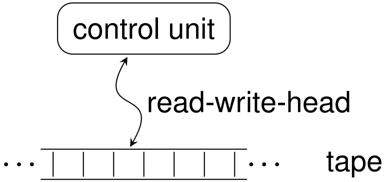

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The tape is two-sided infinite: unlimited memory!

Turing Machines

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We introduce a blank symbol $\Box$. The initial tape content is

$$\cdots \Box\Box\Box\Box \text{ input word } \Box\Box\Box\Box \cdots$$

There is a finite set of states Q and a finite tape alphabet Γ.

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$\delta(q, a) = (q', b, X)$ means: if

- the machine is in state q, and
- the head reads a from the tape

then

- then a is overwritten by b,
- the head moves 1 position left if $X = L$, right if $X = R$, and
- the machine switches to state q'.

Turing Machines

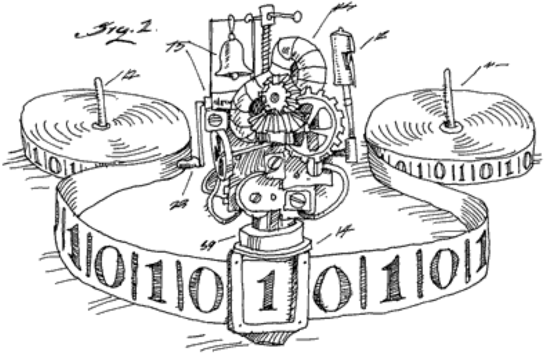
A deterministic Turing machine, short TM, is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$$

where

- Q is a finite set of states,
- $\Sigma \subseteq \Gamma \setminus \{ \Box \}$ a finite input alphabet,
- Γ a finite tape alphabet,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{ L, R \}$ a partial transition function,
- q_0 the starting state,
- $\Box \in \Gamma$ the blank symbol,
- $F \subseteq Q$ a set of final (accepting) states.

Turing Machines



Turing Machine Configuration

A **configuration** of a TM is a word vqw ($q \in Q$, $v, w \in \Gamma^*$).

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Turing Machines and Languages

The **language $L(M)$ accepted by** TM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ is

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So a configuration vqw with $q \in F$ is a halting state.

Note that $w \notin L(M)$ can have two causes:

- the execution halts in a state $q \notin F$, or
- the execution is infinite (never halts).

Example

We construct a TM M with $L(M) = \{ a^n b^n c^n \mid n \geq 1 \}$.

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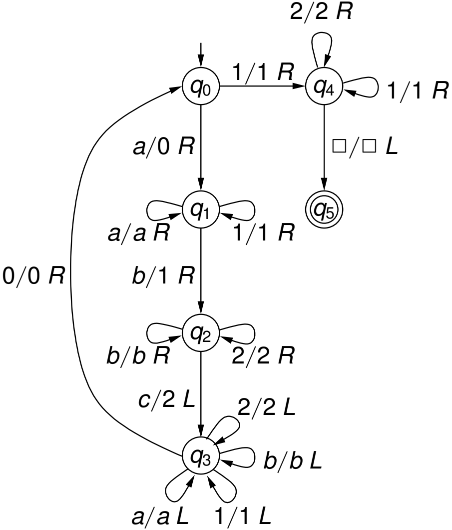
- $\Sigma = \{ a, b, c \}$ and $\Gamma = \{ a, b, c, 0, 1, 2, \Box \}$
- q_0: Read a, replace by 0, move right and switch to q_1.
- q_1: Keep moving right until we read b.
  Replace b by 1, move right and switch to q_2.
- q_2: Keep moving right until we read c.
  Replace c by 2, move left and switch to q_3.
- q_3: Keep moving left until we read 0.
  Move right and switch back to q_0.
- If we read 1 in q_0, switch to q_4.
- q_4: Keep moving right to check whether there are a's, b's or c's left. If not, then go to final state q_5.

Example



2/2 R
1/1 R
q0
q4
1/1 R
a/0 R
□/□ L
q1
q5
a/a R
1/1 R
0/0 R
b/1 R
q2
b/b R
2/2 R
c/2 L
2/2 L
q3
b/b L
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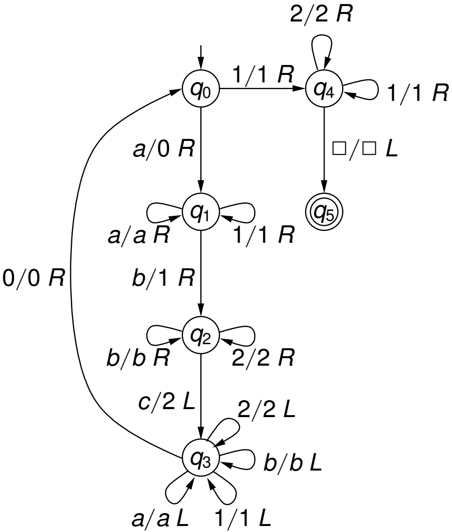

$q_0 aabbcc$

$\vdash^+ 0 q_0 a1b2c$

$\vdash^+ 00 q_0 1122$

$\vdash^+ 00112 q_5 2$

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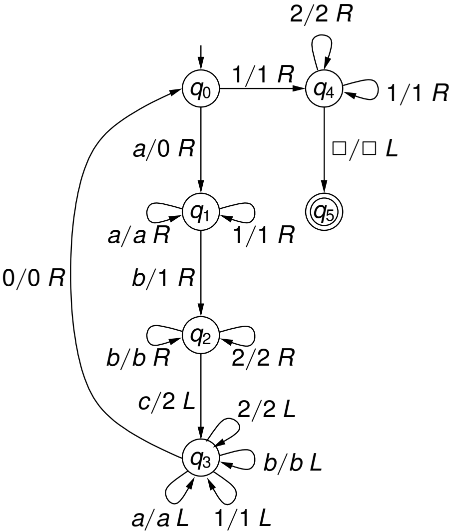

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$\vdash^+ 00 q_0 11b22$

$\vdash^+ 0011 q_4 b22$

Exercise

(Groups of two, 1 minutes)

Construct a Turing machine accepting all words of **odd** length over the alphabet $\Sigma = \{a, b\}$.

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A language L is **recursively enumerable** if L is accepted by a (deterministic) Turing machine.

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Extensions of Turing machines, such as

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Church-Turing thesis: Every computation of a computer can be simulated by a deterministic Turing machine.

Alonzo Church & Alan Turing

Alonzo Church (1903-1995) is inventor of the λ-calculus.

Alan Turing (1912-1954)

- introduced the Turing machine,
- invented the Turing test,
- played a key role in cracking the German Enigma machine.

In 1938, both proved undecidability of validity in predicate logic.

And there is of couse the famous Church-Turing thesis.

Looking Back

- Proving that a language is not context-free
  (Pumping lemma for context-free languages)
- If L_1, L_2 context-free, then
  - $L_1 \cup L_2$, L_1^R, $L_1 L_2$, L_1^* context-free
  - $L_1 \cap L_2$, $\overline{L_1}$, $L_1 \setminus L_2$ not always context-free
- Turing machines: automata with unlimited memory (tape)

Looking Forward

Read:

- Linz 8.1–8.2, 9.1, 9.3

Do the following exercises:

- Linz 8.1: 2, 3, 5, 8c
- Linz 8.2: 1, 11, 22, 23
- Linz 9.1: 4, 5, 7d,e, 8

Following lecture:

- Variations of Turing machines
- Recursive and recursively enumerable languages
- Unrestricted, context-sensitive grammars
- Linear bounded automata