Automata Theory :: Regular Expressions

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Regular Expressions

We define the **regular expressions** over an alphabet Σ:

- $\emptyset$ is a regular expression
- λ is a regular expression
- a is a regular expression for every $a \in \Sigma$
- $r_1 + r_2$ is a regular expression for all regular expr. r_1 and r_2
- $r_1 \cdot r_2$ is a regular expression for all regular expr. r_1 and r_2
- r^* is a regular expression for all regular expressions r

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A regular expression is syntax, describing a language.

Every **regular expression** r defines a **language** $L(r)$:

$$L(\varnothing) = \varnothing \qquad L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(\lambda) = \{\lambda\} \qquad L(r_1 \cdot r_2) = L(r_1)L(r_2)$$

$$L(a) = \{a\} \quad \text{for } a \in \Sigma \qquad L(r^*) = L(r)^*$$

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$L((a + b) \cdot c^*) = (\{a\} \cup \{b\})\{c\}^* = \{a, b\}\{c\}^*$

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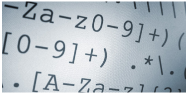
Regular expressions are used to search and manipulate text.

For example:

- `grep` in Linux
- script languages such as Perl



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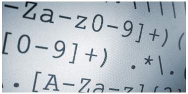

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Find a regular expression r over $\Sigma = \{a, b\}$ such that $L(r)$ consists of all words that contain the pattern *bab*:

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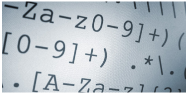

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$$(a + b)^* \cdot b \cdot a \cdot b \cdot (a + b)^*$$

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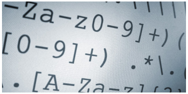

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$$(a+b)^* \cdot b \cdot a \cdot b \cdot (a+b)^* = (a+b)^*\ bab\ (a+b)^*$$

Regular Expressions $\iff$ Regular Languages

Theorem

A language L is **regular**
$\iff$ there is a **regular expression** r with $L(r) = L$.

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Proof.

We need to prove two directions:

- $(\Leftarrow)$ Translate regular expressions into NFAs.
- $(\Rightarrow)$ Translate NFAs into regular expressions.

$\square$

($\Leftarrow$) From Regular Expressions to NFAs

Construction ($\Leftarrow$)

For every regular expression r, we build an NFA M such that

- $L(M) = L(r)$,
- M has precisely one final and one (different) starting state

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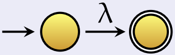

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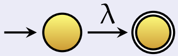

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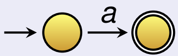

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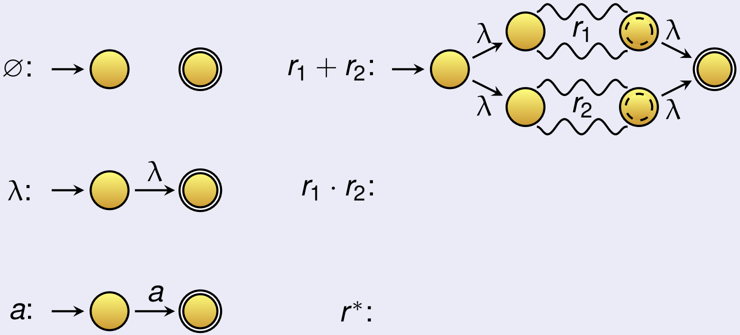

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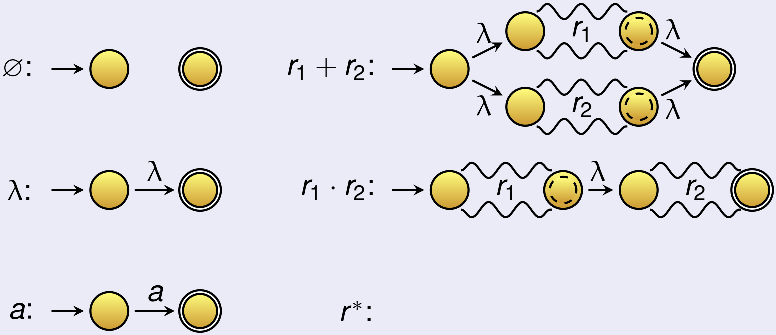

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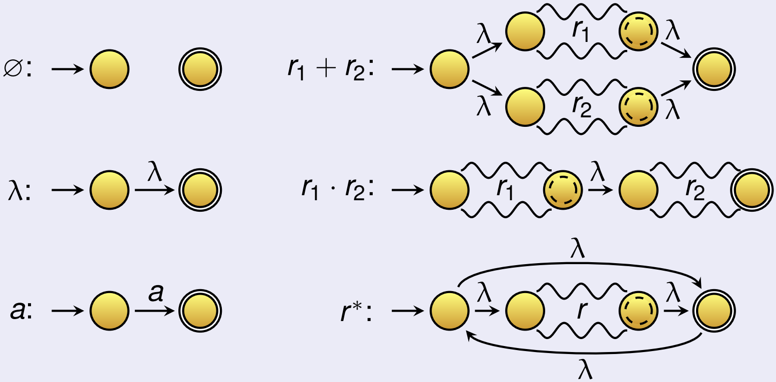
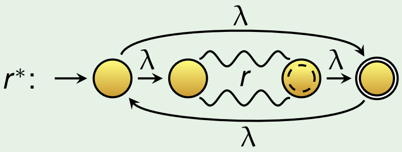


Exercise

Understanding the start case



Note that:

- $((a^*) \cdot b)^*$ shows that the new starting state is needed
- $(a \cdot (b^*))^*$ shows that the new final state is needed

What goes wrong without introducing the new start/final state?

($\Rightarrow$) From NFAs to Regular Expressions (1)

Construction ($\Rightarrow$)

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Step 1:

We transform M such that there is

- precisely one initial state
- precisely one final state

($\Rightarrow$) From NFAs to Regular Expressions (2)

Step 2:

We remove all double arrows.

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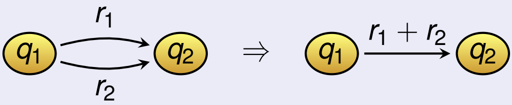
If there are 2 arrows from a state q_1 to q_2 with labels r_1 and r_2 replace them by one arrow with label $r_1 + r_2$:



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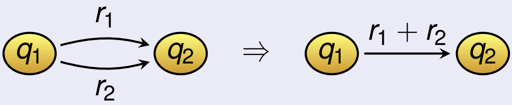

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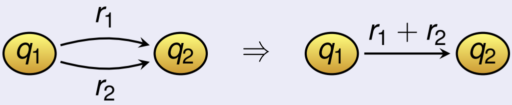

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We remove all double arrows before continuing with Step 3.

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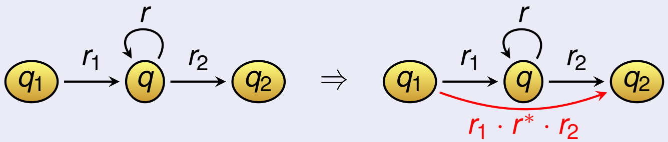

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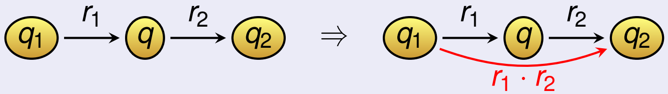

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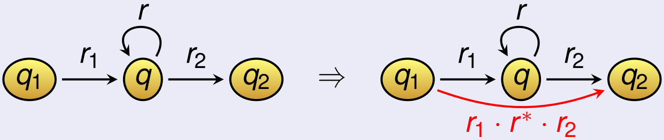

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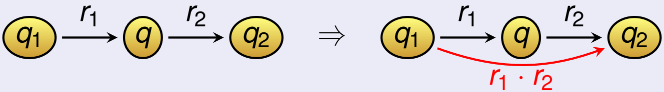

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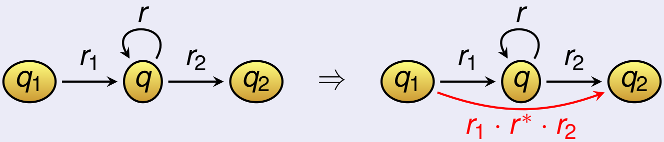

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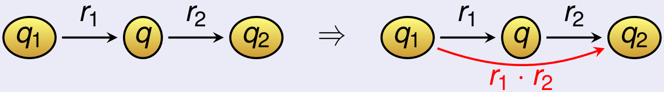

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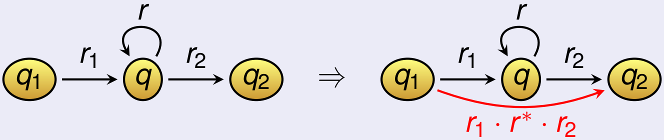

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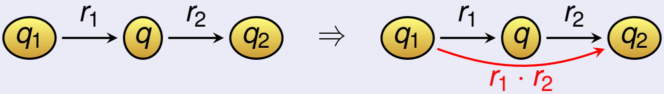

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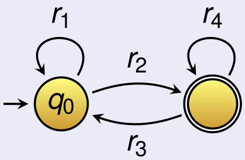
We repeat Step 2 and Step 3 until there is nothing to be done.

($\Rightarrow$) From NFAs to Regular Expressions (4)

Step 4:

If $F \neq \{ q_0 \}$, then the transition graph is finally of the form:



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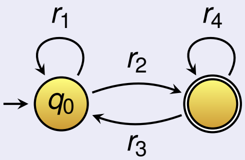

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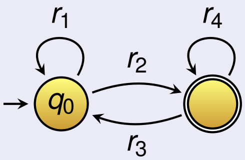

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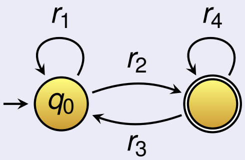

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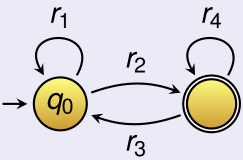

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Question

What is the form of the transition graph and regular expression for the case that $F = \{ q_0 \}$?

Exercise

Find a regular expression r such that

$$L(r) = \{\, w \in \{a, b\}^* \mid n_a(w) \text{ even and } n_b(w) \text{ is odd} \,\}$$

where

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Find a regular expression r over $\{a, b\}$ such that $L(r)$ consists of all words that do not contain the pattern *bab*.