Automata & Complexity

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Looking Back

Previous subjects relevant for this lecture:

- Context-free languages
- Context-free grammars
- Parsing algorithms (CYK and LL)

Pushdown Automata

Goal

A class of automata that accepts the context-free languages.

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A pushdown automaton has a **stack** of unlimited size.

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Initially, stack contains one symbol: **stack start symbol** $z \in \Gamma$.

Pushdown Automata

A **nondeterministic pushdown automaton** (**NPDA**) is a tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

- Q is a finite set of states
- Σ is a finite input alphabet
- Γ is a finite stack alphabet
- $\delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \to 2^{Q \times \Gamma^*}$

  the transition function, where $\delta(q, \alpha, b)$ is always finite
- $q_0 \in Q$ the starting state
- $z \in \Gamma$ the stack starting symbol
- $F \subseteq Q$ a set of final states

Language Accepted by a Pushdown Automaton

When a NPDA reads a words, we need to keep track of:

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The **language generated by** NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is

$$L(M) = \{\, w \in \Sigma^* \mid (q_0, w, z) \vdash^* (q', \lambda, u) \text{ where } q' \in F \,\}.$$

Note: no condition on the stack u at the end. It is often empty.

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Intuition:

- In q_0 a stack $1^k 0$ means: we have read k a's.
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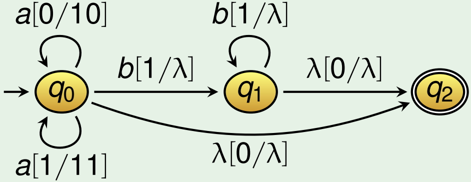
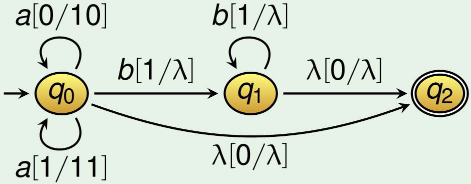


Example Computation



Stepwise reading of the word *aabb*:

$$(q_0, aabb, 0)$$

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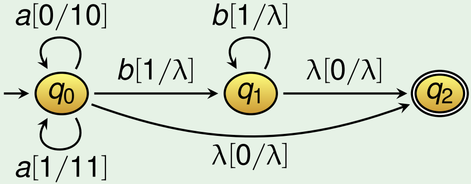

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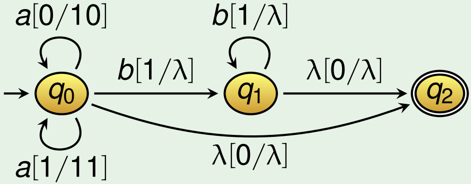

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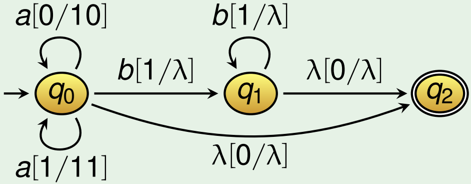

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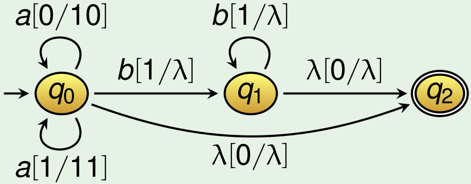

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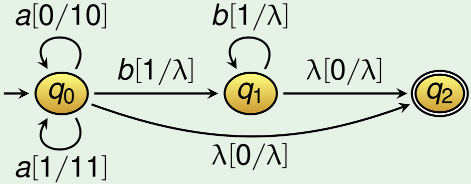

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Exercises

(Groups of two, 2 minutes)

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Hint: define

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Is there an NPDA M with $L(M) = \{ ww \mid w \in \{a, b\}^+ \}$?

Deterministic Pushdown Automaton

A **deterministic pushdown automaton** (**DPDA**) is an NPDA such that

- $\delta(q, \alpha, b)$ contains at most one element
- If $\delta(q, \lambda, b) \neq \emptyset$, then $\delta(q, a, b) = \emptyset$ for every $a \in \Sigma$.

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A language L is **deterministic context-free** if there exists a DPDA M with $L(M) = L$.

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A deterministic context-free language allows for efficient parsing.

Exercises

Which of these languages are deterministic context-free?

- $\{ a^n b^n \mid n \geq 0 \}$
- $\{ ww^R \mid w \in \{a, b\}^+ \}$
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Not all context-free languages are deterministic context-free.

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Conclusion

Not all context-free languages are deterministic context-free.

Theorem

It is decidable if two DPDA's generate the same language.

(Géraud Sénizergues, 1997)

Context-Free Languages and NPDA's

Theorem

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Proof.

We need to prove two directions:

- $(\Rightarrow)$ Translate context-free grammars into NPDA's.
- $(\Leftarrow)$ Translate NPDA's into context-free grammars.

$\square$

From Context-Free Grammars to NPDA's

Construction

Let $G = (V, T, S, P)$ be a context-free grammar.

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We simulate a leftmost reduction:

$$\delta(q_1, \lambda, A) = \{\, (q_1, x) \mid A \to x \in P \,\} \qquad (A \in V)$$
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Then $L(M) = L(G)$.

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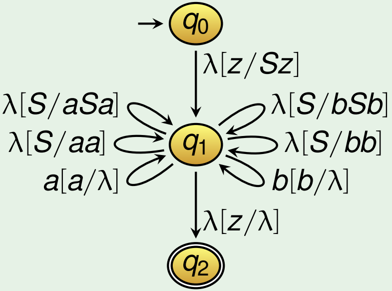

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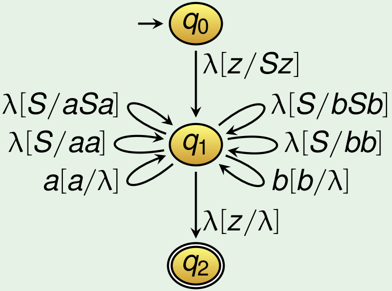

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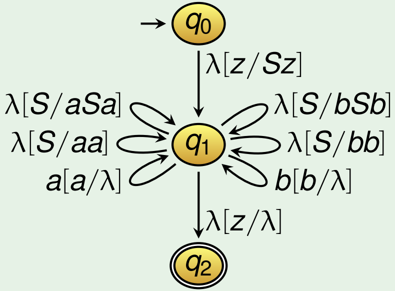

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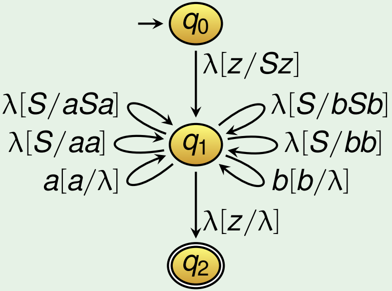

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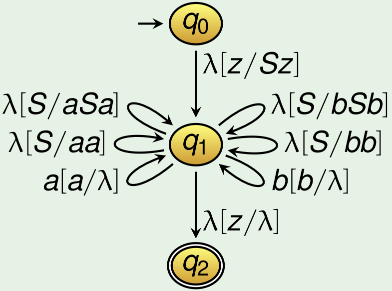

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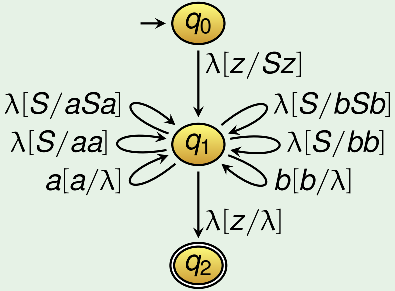

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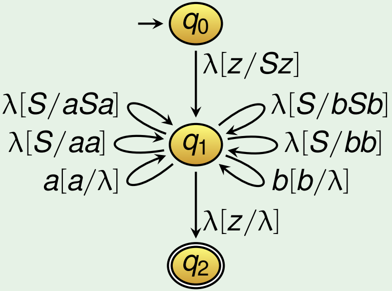

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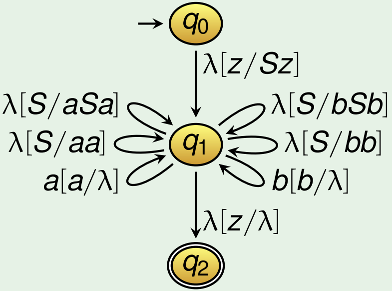

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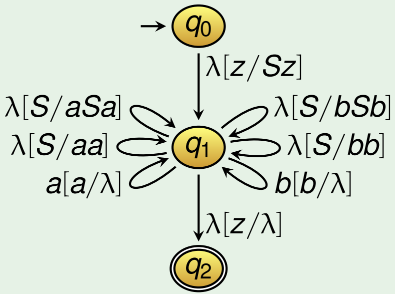

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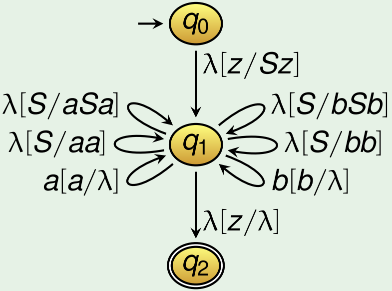

$$(q_0, abba, z) \vdash (q_1, abba, Sz) \vdash (q_1, abba, aSaz) \vdash (q_1, bba, Saz)$$
$$\vdash (q_1, bba, bbaz) \vdash (q_1, ba, baz) \vdash (q_1, a, az)$$
$$\vdash (q_1, \lambda, z) \vdash (q_2, \lambda, \lambda)$$

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How to transform an NPDA into an equivalent NPDA such that

- $F = \{ q_f \}$, and
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(Intuition: switch to final state q_f when stack is empty.)
- Define $\widehat{q_0}$ as starting state and $F = \{q_f\}$.

From NPDA's to Context-Free Grammars

Construction

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ be an NPDA.

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The set P contains the following rules:

- If $(q', \lambda) \in \delta(q, \alpha, b)$, then
  $(q\, b\, q') \to \alpha$ in P.
- If $(q', c_1 \cdots c_n) \in \delta(q, \alpha, b)$ with $n \geq 1$, then
  $(q\, b\, r_n) \to \alpha\, (q'\, c_1\, r_1)\, (r_1\, c_2\, r_2) \cdots (r_{n-1}\, c_n\, r_n)$ in P
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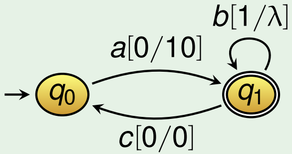
Then we have $L(G) = L(M)$.

Example

Consider the following NPDA with stack starting symbol $z = 0$:



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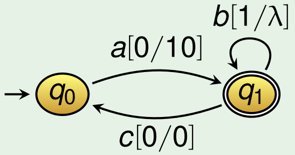

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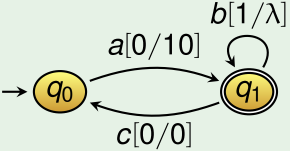

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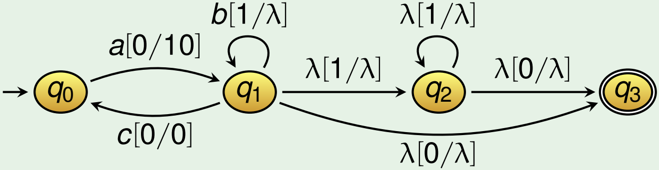

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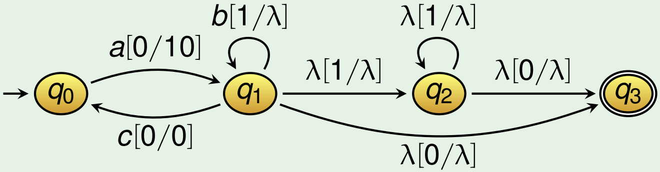
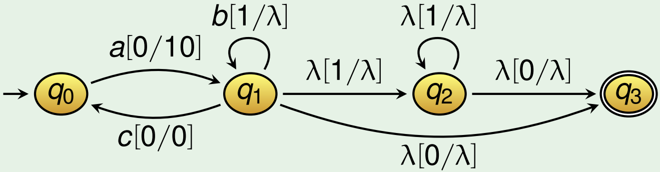


The resulting context-free grammar is:

$(q_0\, 0$

Example

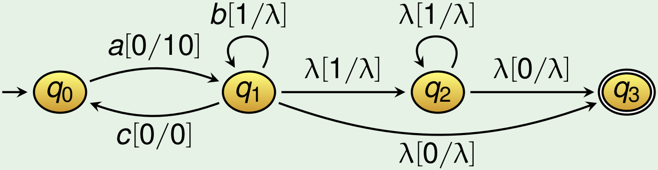


The resulting context-free grammar is:

$$(q_0\,0\,r_2) \to a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2)$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$.

Example



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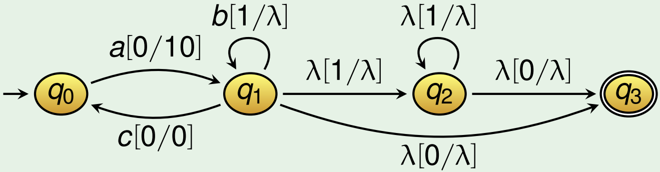

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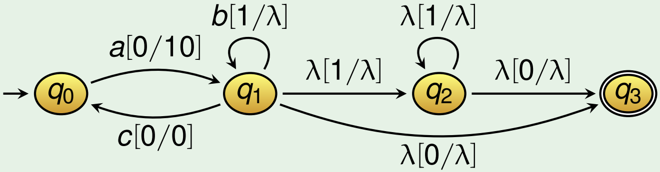

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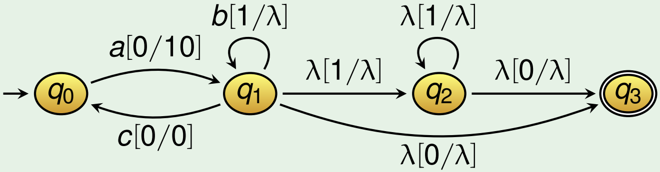

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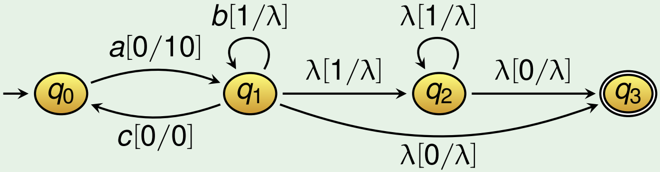

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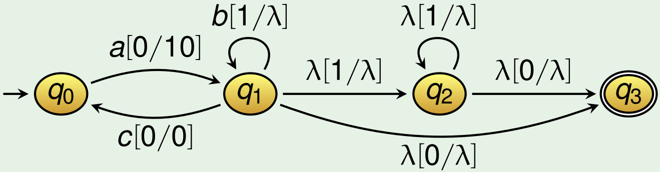

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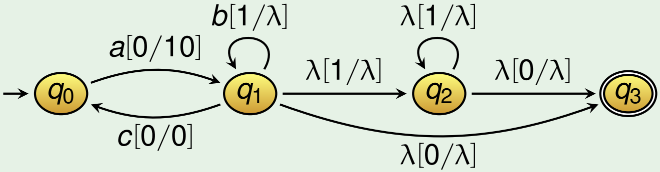

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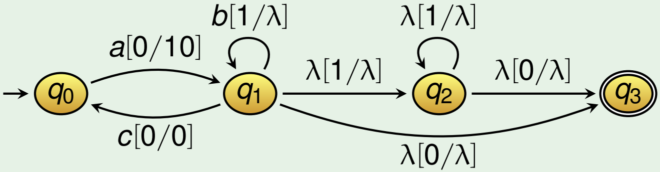

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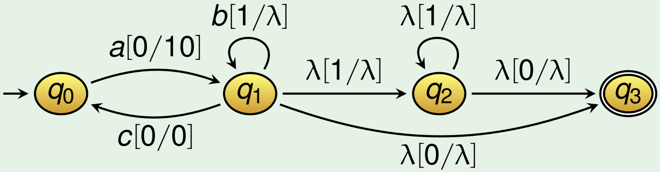

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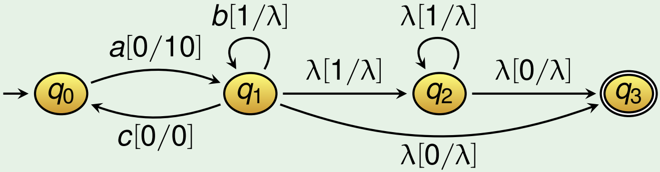

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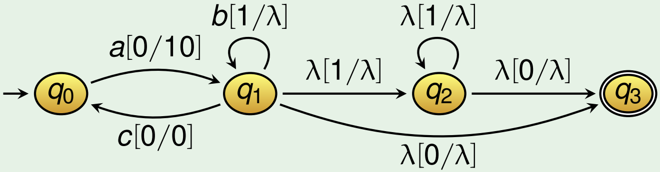

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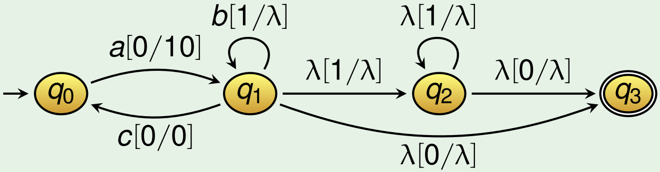

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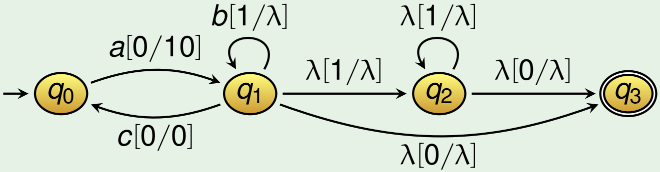

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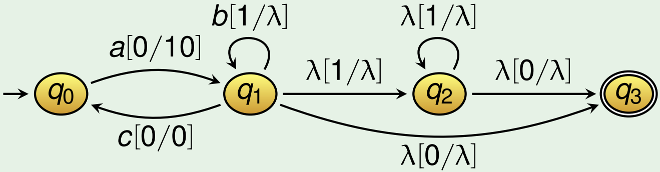

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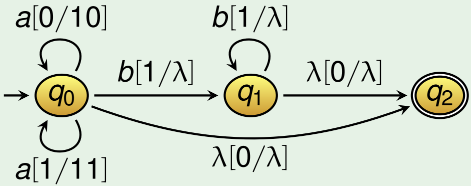
$$(q_0\,0\,q_3) \Rightarrow a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3) \Rightarrow a\,(q_2\,0\,q_3) \Rightarrow a$$

$$(q_0\,0\,q_3) \Rightarrow a\,(q_1\,1\,q_1)\,(q_1\,0\,q_3) \Rightarrow ab\,(q_1\,0\,q_3)$$
$$\Rightarrow abc\,(q_0\,0\,q_3) \Rightarrow^+ abca$$

Exercise

(Groups of two, 2 minutes)

Construct a context-free grammar for the following NPDA.
The stack starting symbol is $z = 0$.



Use the grammar to show that *aabb* is in the language.

Looking Back

- Pushdown automata (with stack)
- Deterministic context-free languages
- Equivalence of NPDA's and context-free grammars

Looking Forward

Read:

- Linz 7.1–7.3
- Lewis & Papadimitriou 3.4 (optional)

Do the following exercises:

- Linz 7.1: 3a, 4c,g, 5, 11
- Linz 7.2: 5, 6, 15
- Linz 7.3: 3, 5, 8, 16, 18

Following lecture:

- Properties of context-free languages
- Turing machines