Automata & Complexity

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Looking Back

Previous subjects relevant for this lecture:

- languages as sets of words
- finite automata: deterministic and non-deterministic
- language L is regular $\iff$ L accepted by a DFA
- language L is regular $\iff$ L accepted by a NFA

Grammars

A **grammar** defines a **language**.

Applications areas:

- natural language,
- artificial intelligence,
- syntax of programming languages.

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Example

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\begin{aligned}
\langle \text{sentence} \rangle &\rightarrow \langle \text{article} \rangle \ \langle \text{noun} \rangle \ \langle \text{verb} \rangle \ \langle \text{article} \rangle \ \langle \text{noun} \rangle \\
\langle \text{article} \rangle &\rightarrow \text{the} \\
\langle \text{article} \rangle &\rightarrow \text{a} \\
\langle \text{noun} \rangle &\rightarrow \text{farmer} \\
\langle \text{noun} \rangle &\rightarrow \text{cow} \\
\langle \text{verb} \rangle &\rightarrow \text{milks}
\end{aligned}
$$

With these **grammar rules** we can construct a $\langle \text{sentence} \rangle$.

Grammars Example

$$
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The farmer milks a cow is a sentence in the language.

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$\langle \text{sentence} \rangle$

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$$\langle \text{sentence} \rangle \Rightarrow \langle \text{article} \rangle \langle \text{noun} \rangle \langle \text{verb} \rangle \langle \text{article} \rangle \langle \text{noun} \rangle$$

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&\Rightarrow \text{the } \langle \text{noun} \rangle \langle \text{verb} \rangle \langle \text{article} \rangle \langle \text{noun} \rangle
\end{aligned}
$$

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&\Rightarrow \text{the farmer milks } \langle \text{article} \rangle \langle \text{noun} \rangle
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&\Rightarrow \text{the farmer } \langle \text{verb} \rangle \langle \text{article} \rangle \langle \text{noun} \rangle \\
&\Rightarrow \text{the farmer milks } \langle \text{article} \rangle \langle \text{noun} \rangle \\
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The farmer milks a cow is a sentence in the language.

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&\Rightarrow \text{the farmer } \langle \text{verb} \rangle \langle \text{article} \rangle \langle \text{noun} \rangle \\
&\Rightarrow \text{the farmer milks } \langle \text{article} \rangle \langle \text{noun} \rangle \\
&\Rightarrow \text{the farmer milks a } \langle \text{noun} \rangle \\
&\Rightarrow \text{the farmer milks a cow}
\end{aligned}
$$

The cow milks the farmer is also a sentence in the language.

Grammars

A **grammar** $G = (V, T, S, P)$ consists of:

- finite set V of **non-terminals** (or **variables**)
- finite set T of **terminals**
- a **start symbol** $S \in V$
- finite set P of **production rules** $x \to y$ where
  - $x \in (V \cup T)^+$ containing at least one symbol from V
  - $y \in (V \cup T)^*$

  (for context-free grammars: $x \in V$)

In the previous example:

- variables: $\langle \text{sentence} \rangle$, $\langle \text{article} \rangle$, $\langle \text{noun} \rangle$, $\langle \text{verb} \rangle$
- terminals: the, a, farmer, cow, milks
- starting symbol: $\langle \text{sentence} \rangle$

Languages Generated by Grammars

If $x \to y$ is a production rule, then we have a **derivation step**

$$uxv \Rightarrow uyv$$

for every $u, v \in (V \cup T)^*$.

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Thus there is a derivation $u \Rightarrow^* v$ if v can be obtained from u by zero or more derivation steps.

The **language generated** by a grammar $G = (V, T, S, P)$ is

$$L(G) = \{ w \in T^* \mid S \Rightarrow^* w \}$$

The language consists of all words that

- contain only terminal letters (no variables), and
- can be derived from the start symbol.

Exercises

$G = (\{S\}, \{a, b\}, S, P)$, where P consists of

$$S \to aSb$$

$$S \to \lambda$$

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Example derivations:

$$S \Rightarrow \lambda$$
$$S \Rightarrow aSb \Rightarrow ab$$

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Example derivations:

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What is the language generated by G as a set?

$$L(G) =$$

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What is the language generated by G as a set?

$$L(G) = \{a^n b^n \mid n \geq 0\}$$

Notational Conventions for Grammars

When defining $G = (V, T, S, P)$ we use

- capital case letters for non-terminals (variables)
- lower case letters for terminals

Often, we only specify the production rules.

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(Groups of two, 1 minute)

$$G_1: \quad S \to Ab \qquad\qquad G_2: \quad S \to aSb$$
$$A \to aAb \qquad\qquad S \to b$$
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$L(G_1) = \{a^n b^{n+1} \mid n \geq 0\}$

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$L(G_1) = \{a^n b^{n+1} \mid n \geq 0\}$ $\qquad$ $L(G_2) =$

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$$L(G_1) = \{a^n b^{n+1} \mid n \geq 0\} \quad = \quad L(G_2) = \{a^n b^{n+1} \mid n \geq 0\}$$

Exercises (1)

(Individual, 1 minute)

Find a grammar G such that

$$L(G) = \{a, b\}^* \{c\} \{b, c\}^*$$

Exercises (2)

A word $w = a_1 a_2 \cdots a_n$ is called **palindrome** if $w = w^R$, that is

$$a_1 a_2 \cdots a_n = a_n \cdots a_2 a_1$$

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Find a grammar G that generates all palindromes over the alphabet $\Sigma = \{a, b\}$. In other words

$$L(G) = \{ w \in \Sigma^* \mid w \text{ is a palindrome} \}$$

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(Groups of two, 3 minutes)

Find a grammar G that generates all non-palindromes over the alphabet $\Sigma = \{a, b\}$. In other words

$$L(G) = \{ w \in \Sigma^* \mid w \text{ is not a palindrome} \}$$

B(ackus) N(aur) F(orm) is a Context-Free Grammar

Non-terminals (variables) are indicated by $\langle$ and $\rangle$.

Example

$$
\begin{array}{rcl}
\langle \text{stm} \rangle & \rightarrow & \langle \text{var} \rangle \ \textbf{:=} \ \langle \text{expr} \rangle \\
\langle \text{stm} \rangle & \rightarrow & \langle \text{stm} \rangle \ \textbf{;} \ \langle \text{stm} \rangle \\
\langle \text{stm} \rangle & \rightarrow & \textbf{begin} \ \langle \text{stm} \rangle \ \textbf{end} \\
\langle \text{stm} \rangle & \rightarrow & \textbf{if} \ \langle \text{cond} \rangle \ \textbf{then} \ \langle \text{stm} \rangle \ \textbf{else} \ \langle \text{stm} \rangle \\
\langle \text{stm} \rangle & \rightarrow & \textbf{while} \ \langle \text{cond} \rangle \ \textbf{do} \ \langle \text{stm} \rangle \\
\langle \text{cond} \rangle & \rightarrow & \cdots \\
\langle \text{var} \rangle & \rightarrow & \cdots \\
\langle \text{expr} \rangle & \rightarrow & \cdots \\
\cdots & \rightarrow & \cdots
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\langle \text{var} \rangle & \to & \cdots \\
\langle \text{expr} \rangle & \to & \cdots \\
\cdots & \to & \cdots
\end{array}
$$

Write $\langle var \rangle ::= x \mid y \mid z$ instead of

$$
\begin{array}{l}
\langle var \rangle \to x \\
\langle var \rangle \to y \\
\langle var \rangle \to z
\end{array}
$$

Right Linear Grammars

A grammar $G = (V, T, S, P)$ is **right linear** if all production rules are of the form

$$A \to uB \qquad \text{or} \qquad A \to u$$

with $A, B \in V$ and $u \in T^*$.

Moreover G is **strictly right linear** if $|u| \leq 1$ (i.e. $u \in (T \cup \{\lambda\})$).

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(Individual, 1 minute)

Construct a right linear grammar G such that

$$L(G) = \{a, b\}^* \{aa\} \{b\}^*$$

(Groups of two, 2 minutes)

Construct a right linear grammar G such that

$$L(G) = \{ab\} \left(\{a\}^* \{cb\}\right)^* \{b\}$$

(Strictly) Right Linear Grammars

Theorem

Let G be a right linear grammar G. There exists a **strictly** right linear grammar H such that $L(G) = L(H)$.

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Let $G = (V, T, S, P)$ be a right linear grammar.

Assume that we have a production rule γ of the form $A \rightarrow u(B)$ with $A, B \in V$ and $u \in T^*$ with $|u| > 1$.

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Then $u = aw$ for some $a \in T$ and $w \in T^+$.

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Let X be a fresh variable (that is, $X \notin (V \cup T)$).

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Construction

Let $G = (V, T, S, P)$ be a right linear grammar.

Assume that we have a production rule γ of the form $A \rightarrow u(B)$ with $A, B \in V$ and $u \in T^*$ with $|u| > 1$.

Then $u = aw$ for some $a \in T$ and $w \in T^+$.

Let X be a fresh variable (that is, $X \notin (V \cup T)$).

We add X to V and split the rule γ into:

$$A \rightarrow aX \qquad\qquad X \rightarrow w(B)$$

Then $A \rightarrow aX \rightarrow aw(B) = u(B)$. It follows $L(G) = L(H)$.

(Strictly) Right Linear Grammars

Theorem

Let G be a right linear grammar G. There exists a **strictly** right linear grammar H such that $L(G) = L(H)$.

Construction

Let $G = (V, T, S, P)$ be a right linear grammar.

Assume that we have a production rule γ of the form $A \rightarrow u(B)$ with $A, B \in V$ and $u \in T^*$ with $|u| > 1$.

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Then $A \rightarrow aX \rightarrow aw(B) = u(B)$. It follows $L(G) = L(H)$.

Repeat splitting until $|u| \leq 1$ for all rules.

Right Linear Grammars $\iff$ Regular Languages

Theorem

Language L is **regular**

$\iff$ there is a **right linear grammar** G with $L(G) = L$

Right Linear Grammars $\iff$ Regular Languages

Theorem

Language L is **regular**
$\iff$ there is a **right linear grammar** G with $L(G) = L$

Proof.

We need to prove two directions:

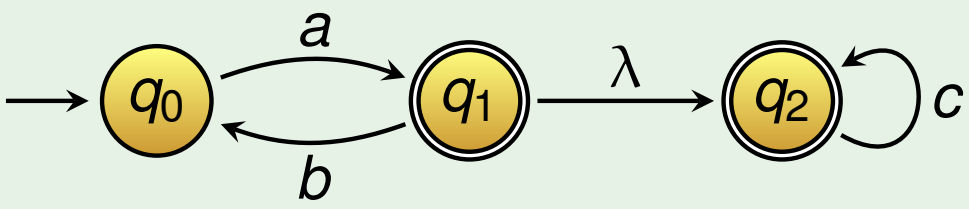
- $(\Rightarrow)$ Translate NFA's (or DFA's) into right linear grammars.
- $(\Leftarrow)$ Translate right linear grammars to NFA's (or DFA's).

$\square$

Exercise for ($\Rightarrow$)

(Groups of two, 2 minutes)

Consider the following NFA M



Construct a right linear grammar G such that:

$$L(M) = L(G)$$

($\Rightarrow$) From NFA's to Right Linear Grammars

Construction ($\Rightarrow$)

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

($\Rightarrow$) From NFA's to Right Linear Grammars

Construction ($\Rightarrow$)

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

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$$L(G) = L(M)$$

We define $V = Q$ and $T = \Sigma$ and $S = q_0$.

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We define $V = Q$ and $T = \Sigma$ and $S = q_0$.

The set P consists of the following production rules

$q \to aq'$	for every $q' \in \delta(q, a)$
$q \to q'$	for every $q' \in \delta(q, \lambda)$
$q \to \lambda$	for every $q \in F$

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There is a derivation in G from S to $u \in T^*$ $\iff$ there is a path in M from S to a state in F labelled with the letters from u.

($\Rightarrow$) From NFA's to Right Linear Grammars

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There is a derivation in G from S to $u \in T^*$ $\iff$ there is a path in M from S to a state in F labelled with the letters from u.

It follows that, $L(G) = L(M)$.

($\Leftarrow$) From Right Linear Grammars to NFA's

Construction ($\Leftarrow$)

Let $G = (V, T, S, P)$ be a right linear grammar.

($\Leftarrow$) From Right Linear Grammars to NFA's

Construction ($\Leftarrow$)

Let $G = (V, T, S, P)$ be a right linear grammar.

Transform G to strictly right linear grammar generating the same language.

($\Leftarrow$) From Right Linear Grammars to NFA's

Construction ($\Leftarrow$)

Let $G = (V, T, S, P)$ be a right linear grammar.

Transform G to strictly right linear grammar generating the same language. Then all rules are of the form

$$A \to uB \qquad \text{or} \qquad A \to u$$

with $A, B \in V$ and $u \in (T \cup \{\lambda\})$.

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We construct an NFA $M = (Q, \Sigma, \delta, q_0, F)$ with $L(M) = L(H)$:

$$\Sigma = T \qquad Q = V \cup \{A_f\} \qquad q_0 = S \qquad F = \{A_f\}$$

where $A_f \notin V$.

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where $A_f \notin V$. We introduce the following arrows in M:

- $A \xrightarrow{u} B$ for every production rule $A \to uB$, and
- $A \xrightarrow{u} A_f$ for every production rule $A \to u$.

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Then $S \Rightarrow^* w \in T^* \iff M$ accepts w.

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We construct an NFA $M = (Q, \Sigma, \delta, q_0, F)$ with $L(M) = L(H)$:

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where $A_f \notin V$. We introduce the following arrows in M:

- $A \xrightarrow{u} B$ for every production rule $A \to uB$, and
- $A \xrightarrow{u} A_f$ for every production rule $A \to u$.

Then $S \Rightarrow^* w \in T^* \iff M$ accepts w. Hence $L(G) = L(M)$.

Exercise

(Groups of two, 2 minutes)

Construct an NFA that accepts the language generated by

$$S \to aT \qquad\qquad T \to abcS \mid b$$

Note that $T \to abS \mid b$ is short for two rules:

$$T \to abcS$$
$$T \to b$$

Left Linear Grammars

A grammar $G = (V, T, S, P)$ is **left linear** if all production rules are of the form

$$A \to Bu \qquad \text{or} \qquad A \to u$$

with $A, B \in V$ and $u \in T^*$.

(Difference with right linear grammars highlighted in red.)

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Theorem

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Proof.

$$\begin{aligned} L \text{ is regular} &\iff L^R \text{ is regular} \\ &\iff \text{right linear grammar for } L^R \\ &\iff \text{left linear grammar for } L \end{aligned}$$

(For the last step, reverse all production rules.) $\square$

Mixing Right and Left Linear Rules

Mixing right **and** left linear rules, the generated language is **not** always regular.

Example

Let G be the grammar

$$S \to aA$$
$$A \to Sb$$
$$S \to \lambda$$

Every rule of G is either right or left linear.

However, the language $L(G) = \{a^n b^n \mid n \geq 0\}$ is **not** regular.

Regular Expressions

We define the **regular expressions** over an alphabet Σ:

- $\emptyset$ is a regular expression
- λ is a regular expression
- a is a regular expression for every $a \in \Sigma$
- $r_1 + r_2$ is a regular expression for all regular expr. r_1 and r_2
- $r_1 \cdot r_2$ is a regular expression for all regular expr. r_1 and r_2
- r^* is a regular expression for all regular expressions r

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A regular expression is syntax, describing a language.

Every **regular expression** r defines a **language** $L(r)$:

$$L(\emptyset) = \emptyset \qquad L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(\lambda) = \{\lambda\} \qquad L(r_1 \cdot r_2) = L(r_1)L(r_2)$$

$$L(a) = \{a\} \quad \text{for } a \in \Sigma \qquad L(r^*) = L(r)^*$$

Example

Example

$L((a+b)\cdot c^*) = \{a,b\}\{c\}^*$

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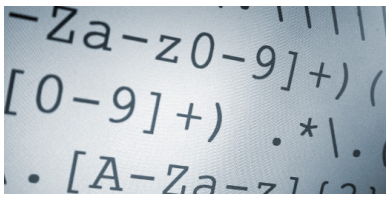
Regular expressions are used to search and manipulate text.

For example:

- `grep` in Linux
- script languages such as Perl



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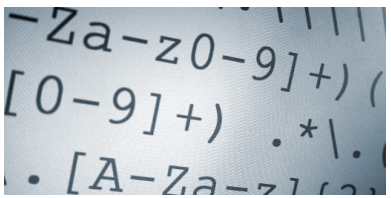

Every major programming language has regular expressions.

(Groups of two, 1 minute)

Exercise

Find a regular expression r over $\Sigma = \{a,b\}$ such that $L(r)$ consists of all words that contain the pattern *bab*.

Regular Expressions $\iff$ Regular Languages

Theorem

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Regular Expressions $\iff$ Regular Languages

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A language L is **regular**
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Proof.

We need to prove two directions:

- ($\Leftarrow$) Translate regular expressions into NFA's.
- ($\Rightarrow$) Translate NFA's into regular expressions.

$\square$

($\Leftarrow$) From Regular Expressions to NFA's

Construction ($\Leftarrow$)

For every regular expression r, we build an NFA M such that

- $L(M) = L(r)$,
- M has precisely one final state (different from staring state)

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We construct M by induction (recursion) on r.

$\emptyset$:

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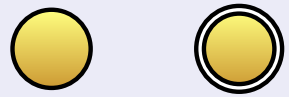
$\emptyset$: 

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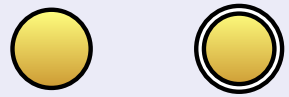
$\emptyset$: 

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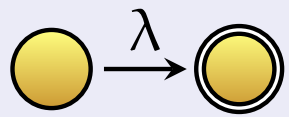
λ: 

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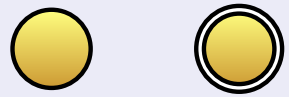
$\emptyset$: 

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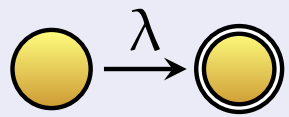
λ: 

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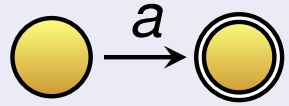
a: 

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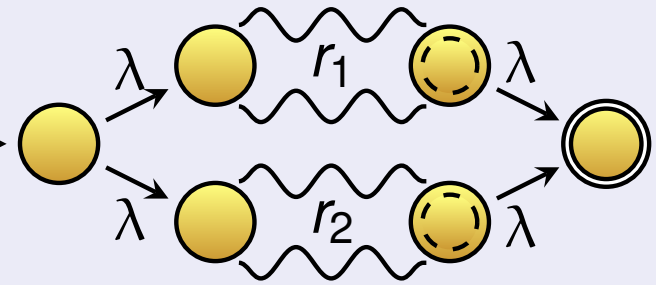
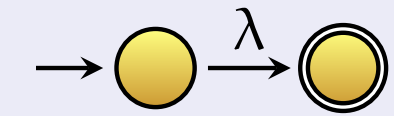
$r_1 + r_2$: 

λ: 

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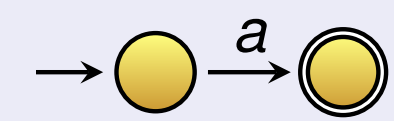
a: 

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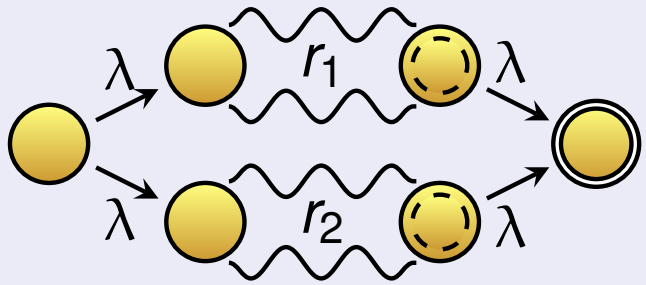
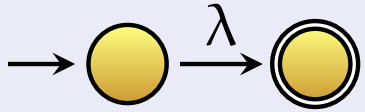
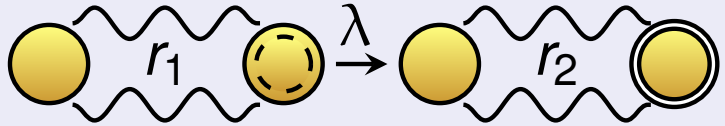
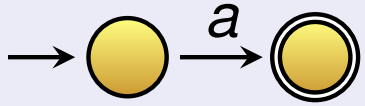
$r_1 + r_2$: 

λ: 

$r_1 \cdot r_2$: 

a: 

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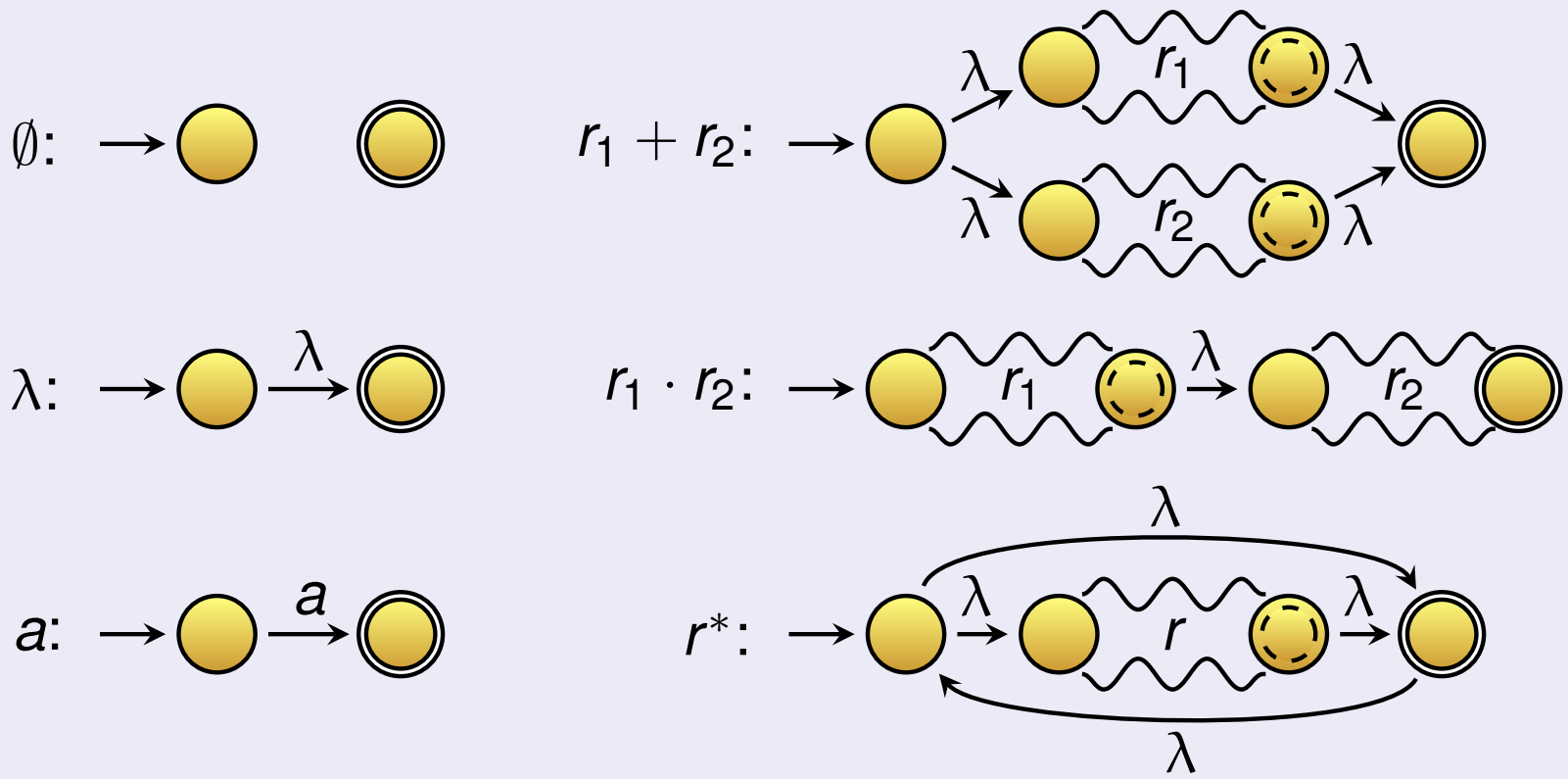
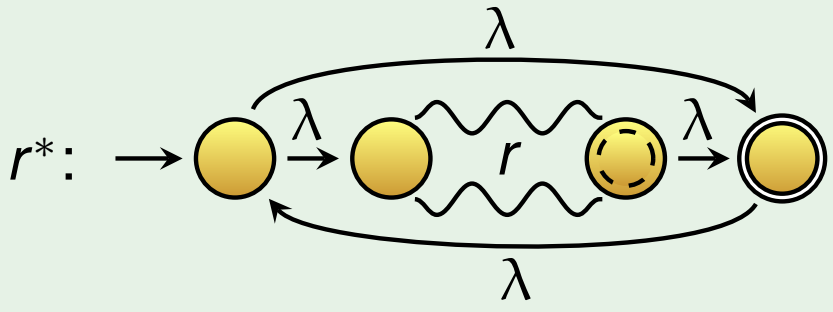


Exercise

Understanding the start case



Note that:

- $((a^*) \cdot b)^*$ shows that the new starting state is needed
- $(a \cdot (b^*))^*$ shows that the new final state is needed

What goes wrong without introducing the new start/final state?

($\Rightarrow$) From NFA's to Regular Expressions (1)

Construction ($\Rightarrow$)

For every NFA M, we construct a regular expression r with

$$L(r) = L(M)$$

($\Rightarrow$) From NFA's to Regular Expressions (1)

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We transform M such that there is only one final state.

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If M has multiple final states:

- Add a fresh state q_f to M.
- For every final state q in M, add an arrow $q \xrightarrow{\lambda} q_f$ to M.
- Define q_f as the only final state.

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Now M has precisely one final state ($\#F = 1$).

($\Rightarrow$) From NFA's to Regular Expressions (2)

Construction ($\Rightarrow$ continued)

Step 2:

We remove all double arrows.

($\Rightarrow$) From NFA's to Regular Expressions (2)

Construction ($\Rightarrow$ continued)

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We remove all double arrows.

We use transition graphs with regular expressions as labels.

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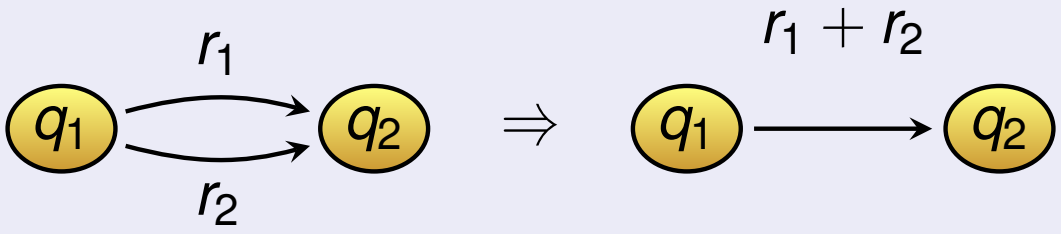

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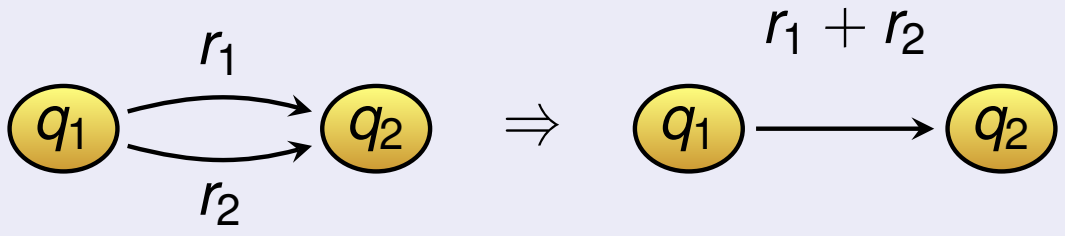
We use transition graphs with regular expressions as labels.

If there are 2 arrows from a state q_1 to q_2 with labels r_1 and r_2 replace them by one arrow with label $r_1 + r_2$:



Note that q_1 can be equal to q_2. Then the arrows are loops!

($\Rightarrow$) From NFA's to Regular Expressions (2)

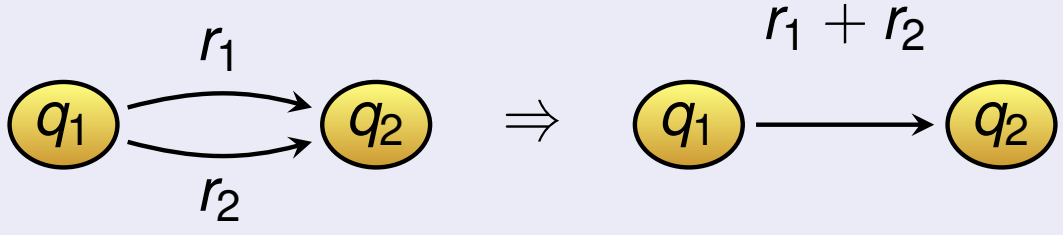
Construction ($\Rightarrow$ continued)

Step 2:

We remove all double arrows.

We use transition graphs with regular expressions as labels.

If there are 2 arrows from a state q_1 to q_2 with labels r_1 and r_2 replace them by one arrow with label $r_1 + r_2$:



Note that q_1 can be equal to q_2. Then the arrows are loops!

We remove all double arrows before continuing with Step 3.

($\Rightarrow$) From NFA's to Regular Expressions (3)

Construction ($\Rightarrow$ continued)

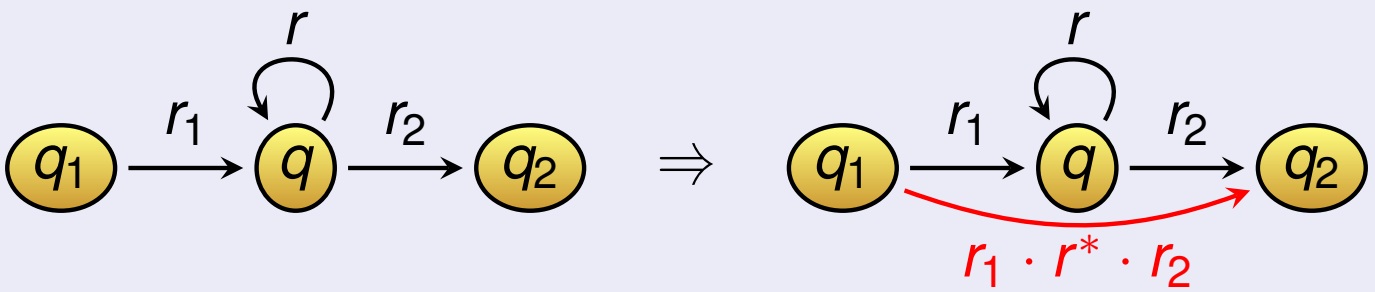
Step 3: As long as there are states other than the starting and final state, remove one other state q as follows.

(⇒) From NFA's to Regular Expressions (3)

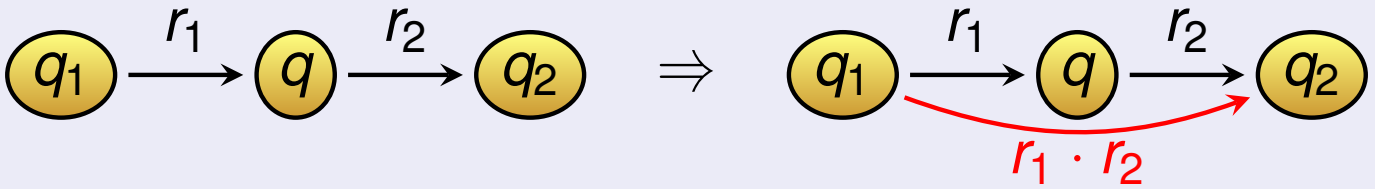
Construction (⇒ continued)

Step 3: As long as there are states other than the starting and final state, remove one other state q as follows.

For all states q_1 and q_2 and arrows $q_1 \xrightarrow{r_1} q$ and $q \xrightarrow{r_2} q_2$, we add an arrow from q_1 to q_2 as follows:



for the case that there is an arrow $q \xrightarrow{r} q$, and otherwise:

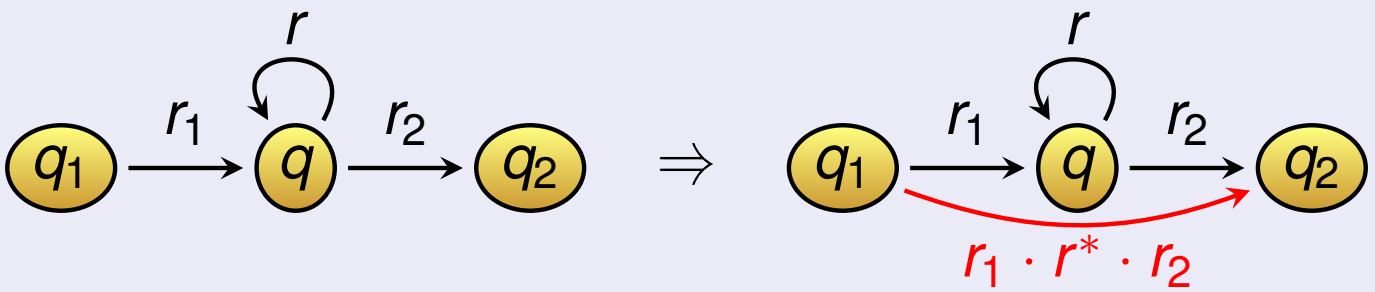


(⇒) From NFA's to Regular Expressions (3)

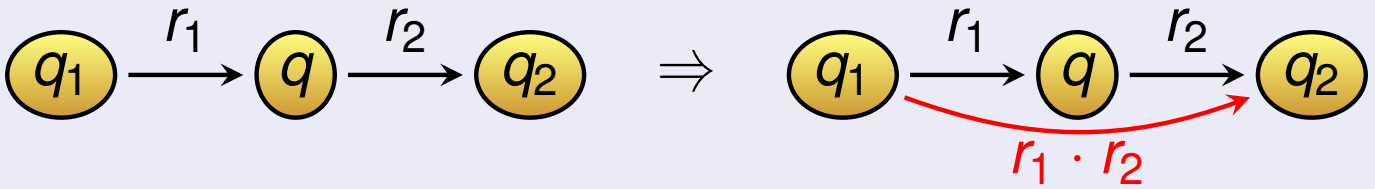
Construction (⇒ continued)

Step 3: As long as there are states other than the starting and final state, remove one other state q as follows.

For all states q_1 and q_2 and arrows $q_1 \xrightarrow{r_1} q$ and $q \xrightarrow{r_2} q_2$, we add an arrow from q_1 to q_2 as follows:



for the case that there is an arrow $q \xrightarrow{r} q$, and otherwise:



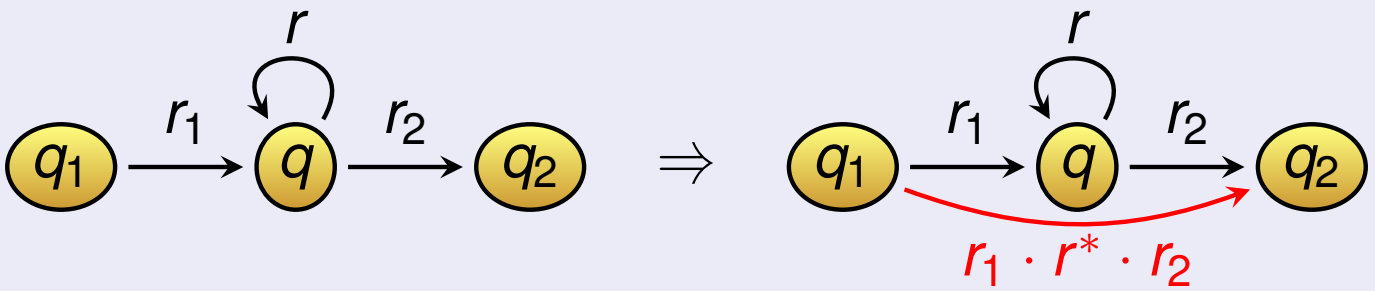
Note that q_1 can be equal to q_2.

($\Rightarrow$) From NFA's to Regular Expressions (3)

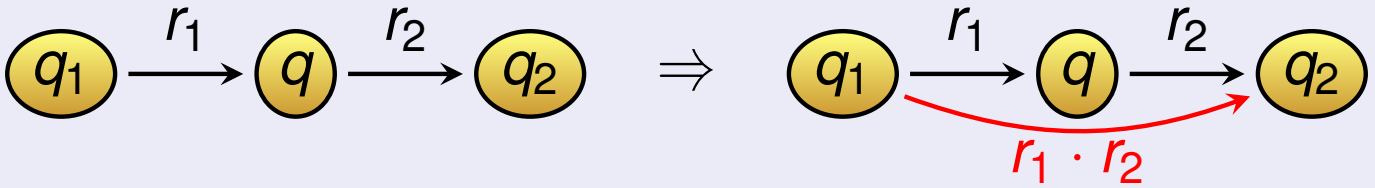
Construction ($\Rightarrow$ continued)

Step 3: As long as there are states other than the starting and final state, remove one other state q as follows.

For all states q_1 and q_2 and arrows $q_1 \xrightarrow{r_1} q$ and $q \xrightarrow{r_2} q_2$, we add an arrow from q_1 to q_2 as follows:



for the case that there is an arrow $q \xrightarrow{r} q$, and otherwise:



Note that q_1 can be equal to q_2.

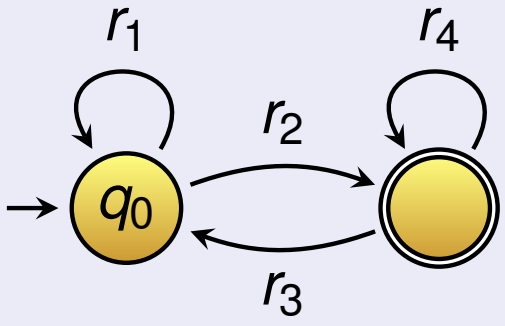
We repeat Step 2 and Step 3 until there is nothing to be done.

(⇒) From NFA's to Regular Expressions (4)

Construction (⇒ continued)

Step 4:

If $F \neq \{q_0\}$, then the transition graph is finally of the form:

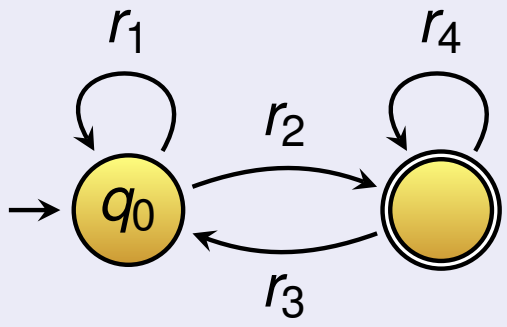


($\Rightarrow$) From NFA's to Regular Expressions (4)

Construction ($\Rightarrow$ continued)

Step 4:

If $F \neq \{q_0\}$, then the transition graph is finally of the form:



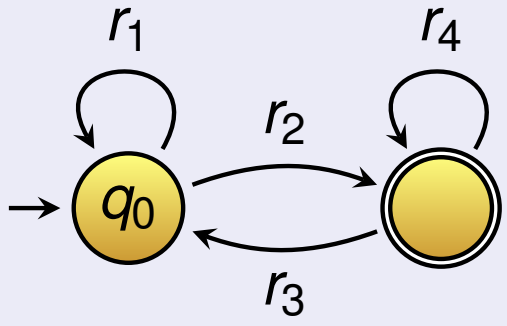
Possibly r_1, r_2, r_3 or r_4 are equal to $\emptyset$.

($\Rightarrow$) From NFA's to Regular Expressions (4)

Construction ($\Rightarrow$ continued)

Step 4:

If $F \neq \{q_0\}$, then the transition graph is finally of the form:



Possibly r_1, r_2, r_3 or r_4 are equal to $\emptyset$.

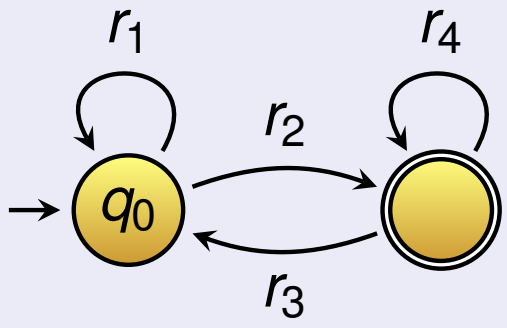
Then the regular expression is:

($\Rightarrow$) From NFA's to Regular Expressions (4)

Construction ($\Rightarrow$ continued)

Step 4:

If $F \neq \{q_0\}$, then the transition graph is finally of the form:



Possibly r_1, r_2, r_3 or r_4 are equal to $\emptyset$.

Then the regular expression is:

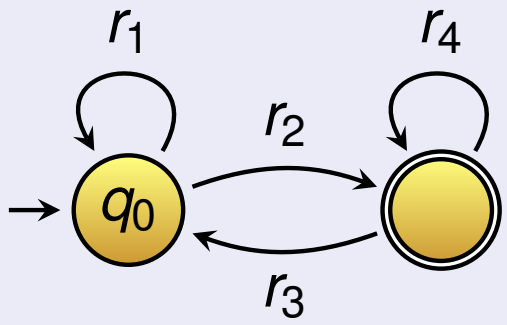
$$L(r_1^* \cdot r_2 \cdot (r_4 + r_3 \cdot r_1^* \cdot r_2)^*) \;=\; L(M)$$

($\Rightarrow$) From NFA's to Regular Expressions (4)

Construction ($\Rightarrow$ continued)

Step 4:

If $F \neq \{q_0\}$, then the transition graph is finally of the form:



Possibly r_1, r_2, r_3 or r_4 are equal to $\emptyset$.

Then the regular expression is:

$$L(r_1^* \cdot r_2 \cdot (r_4 + r_3 \cdot r_1^* \cdot r_2)^*) \;=\; L(M)$$

Question

What is the form of the transition graph and regular expression for the case that $F = \{q_0\}$?

Exercise

(Groups of two, 3 minutes)

Find a regular expression r such that

$$L(r) = \{\, w \in \{a, b\}^* \mid n_a(w) \text{ even and } n_b(w) \text{ is odd} \,\}$$

where

- $n_a(w)$ is the number of a's in w, and
- $n_b(w)$ is the number of b's in w.

Exercise

(Groups of two, 3 minutes)

Find a regular expression r such that

$$L(r) = \{\, w \in \{a, b\}^* \mid n_a(w) \text{ even and } n_b(w) \text{ is odd} \,\}$$

where

- $n_a(w)$ is the number of a's in w, and
- $n_b(w)$ is the number of b's in w.

(Groups of two, 4 minutes)

Find a regular expression r over $\{a, b\}$ such that
$L(r)$ consists of all words that do not contain the pattern *bab*.

Alternative Descriptions of Regular Languages

The following statements are equivalent:

- There is a DFA M with $L(M) = L$.
- There is an NFA M with $L(M) = L$.
- There is a right linear grammar G with $L(G) = L$.
- There is a left linear grammar G with $L(G) = L$.
- There is a regular expression r with $L(r) = L$.

Looking Forward

Read:

- Linz 1.2, 3.1–3.3

Do the following exercises:

- Linz 1.2: 11a,b,c, 13, 14a,b,e,f,h
- Linz 3.1: 5, 7, 13, 16,a,b
- Linz 3.2: 1, 2, 4b,d, 9, 10a,c, 13a
- Linz 3.3: 1, 3, 6, 12

Following lecture:

- Properties of regular languages