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Degrees of Infinite Words, Polynomials, and Atoms*

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We study finite-state transducers and their power for transforming infinite words. Infinite sequences of symbols are of paramount importance in a wide range of fields, from formal languages to pure mathematics and physics. While finite automata for recognising and transforming languages are well-understood, very little is known about the power of automata to transform infinite words.

The word transformation realised by finite-state transducers gives rise to a complexity comparison of words and thereby induces equivalence classes, called *(transducer) degrees*, and a partial order on these degrees. The ensuing hierarchy of degrees is analogous to the recursion-theoretic *degrees of unsolvability*, also known as *Turing degrees*, where the transformational devices are Turing machines. However, as a complexity measure, Turing machines are too strong: they trivialise the classification problem by identifying

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all computable words. Finite-state transducers give rise to a much more fine-grained, discriminating hierarchy. In contrast to Turing degrees, hardly anything is known about transducer degrees, in spite of their naturality.

We use methods from linear algebra and analysis to show that there are infinitely many atoms in the transducer degrees, that is, minimal non-trivial degrees.

Keywords: infinite words; finite-state transducers; degrees of infinite words.

1. Introduction

In recent times, computer science, logic and mathematics have extended the focus of interest from finite data types to include infinite data types, of which the paradigm notion is that of infinite sequences of symbols, or *words*. Infinite words are of paramount importance in a wide range of fields, from formal languages to pure mathematics and physics: they appear in functional programming, formal language theory, in the mathematics of dynamical systems, fractals and number theory, in business (financial data words) and in physics (signal processing).

An accepted and deep mathematical insight is that together with a class of structures, one has to deal with the ways to transform these structures into each other, such as morphisms[a] in a category. Our objects of interest are words and how they can be transformed into each other via finite-state transducers.

While finite automata for recognising and transforming languages are well-studied and well-understood, surprisingly, very little is known about the power of finite automata for transforming words. Even for concrete examples of words w_1 and w_2, there exist no techniques to determine whether w_1 can be transformed into w_2 by some finite-state transducer. See, e.g., Questions Q4 and Q5. We are interested in understanding the power of finite-state transducers for transforming words.

Such a study can be profitably cast in the form of setting up a hierarchy of degrees, induced by a transformational device (sometimes called a 'reduction'). This is a well-known reasoning framework in logic and computer science [13], with many instances, e.g., Wadge degrees [19], Turing degrees [18, 14], r.e. degrees [14], and so on. In our case the hierarchy of degrees is obtained as follows.

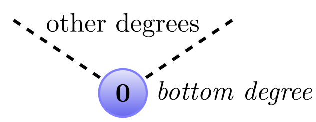
The transformation realised by finite-state transducers induces a partial order of degrees of infinite words: for words $v, w \in \Delta^{\mathbb{N}}$, we write $v \geq w$ if v can be transformed into w by some finite-state transducer. If $v \geq w$, then v can be thought of as *at least as complex as* w. This complexity comparison induces equivalence classes of words, called *degrees*, and a partial order on these degrees, that we call *transducer degrees*.

The ensuing hierarchy of degrees is analogous to the recursion-theoretic *degrees of unsolvability*, also known as *Turing degrees*, where the transformational devices are Turing machines. The Turing degrees have been widely studied in the 60's and 70's. However, as a complexity measure, Turing machines are too strong: they trivialise the classification problem by identifying all computable infinite words. Finite-state transducers (FSTs) give rise to a much more fine-grained hierarchy.

[a]Morphisms in a category should not be confused with morphisms in the sense of this paper.

In our view, transducers are the most natural devices for transforming words. Unlike Turing machines, they are not too strong, but they are still very expressive. On the one hand, transducers are 'weak enough' to exhibit a rich structure within the computable words. On the other hand, they capture several usual transformations, such as alphabet renaming, insertion and removal of elements, or morphisms as usually studied in theories of infinite sequences [1].

Like the Turing degrees, the transducer degrees have a bottom degree that is less than or equal to all other degrees (pictorial on the right). The bottom degree of the Turing degrees contains all computable words. In contrast, transducer degrees are much more fine-grained. The *bottom degree* $\mathbf{0}$ of the transducer degrees consists only of the ultimately periodic words, that is, words of the form $uvvv\cdots$ for finite words u, v.

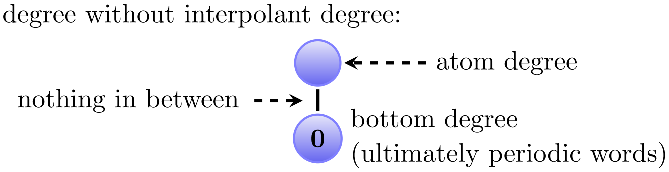


We present a comparison of some basic properties, as to their validity in the Turing degrees and the transducer degrees. For Turing degrees we have [18, 14]

(i) There exist $2^{\aleph_0}$ atom (minimal) degrees. ?
(ii) Every degree has a minimal cover. ?
(iii) Every finite set of degrees has a supremum. ✗
(iv) No infinite ascending sequence has a supremum. ?
(v) There are pairs of degrees without infimum. ✔
(vi) For every degree $\neq \mathbf{0}$ there exists an incomparable degree. ✔
(vii) Every countable partial order can be embedded. ?
(viii) The recursively enumerable degrees are dense. ✗
(ix) The first-order theory of Turing degrees in the language $\langle \geq, = \rangle$ is many-one reducible to that of true second-order arithmetic. ?

Here the symbols on the right indicate whether the properties also hold for transducer degrees: ✔ if the property also holds for transducer degrees, ✗ if it fails, and ? for questions that are open in transducer degrees.

In previous papers [7, 6, 5, 10], we have discussed several structural properties of the hierarchy of transducer degrees. In this paper, we focus on atom degrees. An *atom degree* is a minimal non-trivial degree, that is, a degree that is directly above the bottom degree without interpolant degree:



Thus the atom degrees reduce only to $\mathbf{0}$ or themselves. The following questions are still open for transducer degrees: Q1 Are there $2^{\aleph_0}$ atoms in the transducer degrees? Q2 Do uncomputable atoms exist in the transducer degrees? Q3 Is the degree of

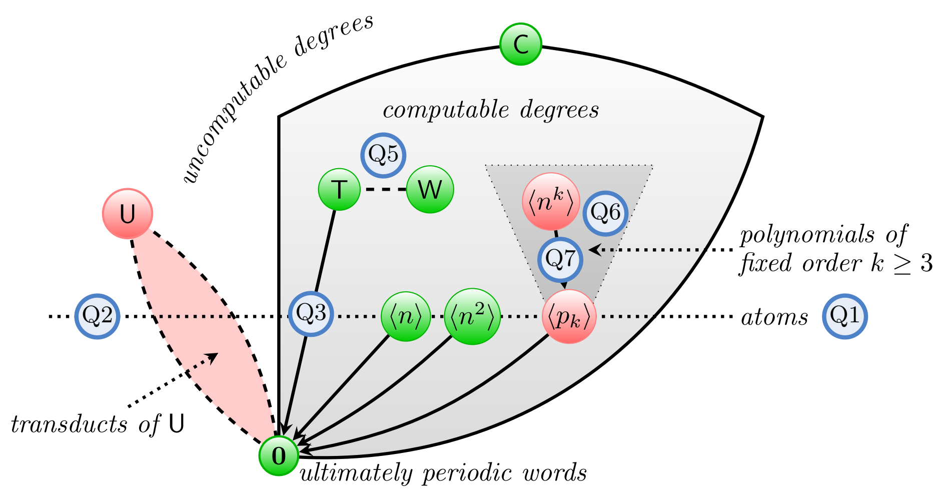


Fig. 1: *The partial order of transducer degrees with focus on the properties studied in this paper. Our contribution is indicated using the colour red. Here p_k is a particular polynomial of order k, see Section 6. The degree of $\langle p_k \rangle$ is an atom and all other polynomials of order k can be transduced to $\langle p_k \rangle$. For $k \geq 3$, the degree $\langle n^k \rangle$ is not an atom as shown in Section 5. The definitions of the words Thue–Morse* T *and the Mephisto Walz* W *are given in Section 2. The degree of* C *is the top degree of the computable words. Finally, the nodes Q1,...,Q7 indicate open problems discussed in Sections 1 and 3.*

the Thue–Morse word $\mathsf{T} = 0110100110010110 \cdots$ an atom?

We show that there are at least $\aleph_0$ atoms residing in the interesting subclass of words that we call *sporadic words*, of which the simplest one is $1\,10\,100\,1000\,10000\,\cdots$. (Jacobs [12] called this word 'rarefied ones'.) Here 'sporadic' refers to the fact that the ones are becoming more and more sporadic. In general, they are of the form $\langle f \rangle = 10^{f(0)}\,10^{f(1)}\,10^{f(2)}\,\cdots$, for some $f : \mathbb{N} \to \mathbb{N}$. This paper studies in particular the case where f is a polynomial. We consider the 'atomicity' of these words depending on the polynomials determining how the ones become ever more sporadic.

Our contribution. The words $\langle n \rangle$ and $\langle n^2 \rangle$ are atoms [7, 5]. Surprisingly, we find that this does not hold for $\langle n^3 \rangle$. In particular, we show that the degree of $\langle n^k \rangle$ is *never* an atom for $k \geq 3$ (see Theorem 23). On the other hand, we prove that for every $k > 0$, there exists a unique atom among the degrees of words $\langle p(n) \rangle$ for polynomials $p(n)$ of order k (see Theorem 32). (To avoid confusion between two meanings of *degrees*, namely *degrees of words* and *degrees of polynomials*, we speak of the *order* of a polynomial.) We moreover show that this atom is the infimum of all degrees of polynomials $p(n)$ of order k. Figure 1 summarises the state of affairs as in this paper. Finally, we show that there exists an uncomputable word U that transduces only to uncomputable words or to ultimately periodic words.

Further related work. Löwe [13] discussed complexity hierarchies derived from notions of reduction. The paper [10] gives an overview over the subject of transducer degrees and compares them with the well-known Turing degrees [18, 14]. Restricting the transducers to output precisely one letter in each step, we arrive at Mealy machines. These give rise to an analogous hierarchy of Mealy degrees that has been studied in [2, 15]. The structural properties of this hierarchy are very different from the transducer degrees [10]. The paper [4] studies a hierarchy of two-sided infinite sequences arising from the transformation realised by permutation transducers.

The current paper is an extension of [8]. Beyond better exposition, more examples and more detailed proofs, the results in Section 7 are new.

2. Preliminaries

Let Σ be an alphabet. The empty word is denote by ε. Let Σ^* be the set of finite words over Σ, and $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. The set of infinite words over Σ is $\Sigma^{\mathbb{N}} = \{\sigma \mid \sigma : \mathbb{N} \to \Sigma\}$ and we let $\Sigma^\infty = \Sigma^* \cup \Sigma^{\mathbb{N}}$. Let $u, w \in \Sigma^\infty$. Then u is called a *prefix of* w, denoted $u \sqsubseteq w$, if $u = w$ or there exists $u' \in \Sigma^\infty$ such that $uu' = w$.

Of particular importance are *morphic words* [1]. For example:

(i) The Thue–Morse word T arises by starting from the word 0, as the limit of repeatedly applying the morphism $0 \mapsto 01$, $1 \mapsto 10$. We abbreviate this by: $\langle 0 \mid 0 \mapsto 01,\ 1 \mapsto 10 \rangle$. The first iterations are $0 \mapsto 01 \mapsto 0110 \mapsto \cdots$.

(ii) The period-doubling word $\mathsf{P} = \langle 0 \mid 0 \mapsto 01,\ 1 \mapsto 00 \rangle$.

(iii) The Mephisto Waltz word $\mathsf{W} = \langle 0 \mid 0 \mapsto 001,\ 1 \mapsto 110 \rangle$.

For a more formal metric definition of morphic words, see [17, 1].

A *sequential finite-state transducer* (FST) [1, 16], a.k.a. *deterministic generalised sequential machine (DGSM)*, is a finite automaton with input letters and finite output words along the edges. A transducer reads the input word letter by letter, in each step producing an output word and changing its state. Then the output word is the concatenation of all the output words encountered along the edges.

Definition 1. A *sequential finite-state transducer* $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ consists of a finite *input alphabet* Σ, a finite *output alphabet* Γ, a finite set of *states* Q, an *initial state* $q_0 \in Q$, a *transition function* $\delta : Q \times \Sigma \to Q$, and an *output function* $\lambda : Q \times \Sigma \to \Gamma^*$. Whenever the alphabets Σ and Γ are clear from the context, we write $A = \langle Q, q_0, \delta, \lambda \rangle$.

An example of an FST is depicted in Figure 2, where we write '$a \,|\, w$' along the transitions to indicate that the input letter is a and the output word is w.

The output given by a transition is allowed to be a *word* over the output alphabet, and not just a single letter or the empty word ε, although that may also be the case. Thereby finite-state transducers generalise the class of Mealy machines that output precisely one letter in each step.

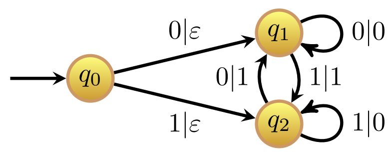


Fig. 2: An FST realising the difference of consecutive bits modulo 2. For example, $\mathsf{T} = 01101001\cdots$ is transformed in $\overline{\mathsf{P}} = 1011101\cdots$ where the overbar signifies inversion between 0 and 1.

We only consider sequential transducers and will simply speak of finite-state transducers henceforth.

Definition 2. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite-state transducer. We homomorphically extend the transition function δ to $Q \times \Sigma^* \to Q$ as follows: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^*$ let $\delta(q, \varepsilon) = q$ and $\delta(q, au) = \delta(\delta(q, a), u)$. We extend the output function λ to $Q \times \Sigma^\infty \to \Gamma^\infty$ as follows: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^\infty$, let $\lambda(q, \varepsilon) = \varepsilon$ and $\lambda(q, au) = \lambda(q, a) \cdot \lambda(\delta(q, a), u)$.

3. Transducer Degrees

We now explain how FSTs give rise to a hierarchy of degrees of infinite words, called transducer degrees. First, we formally introduce the transducibility relation $\geq$ on words as realised by FSTs.

Definition 3. Let $w \in \Sigma^{\mathbb{N}}$, $u \in \Gamma^{\mathbb{N}}$ for finite alphabets Σ, Γ. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a FST. We write $w \geq_{\mathrm{A}} u$ if $u = \lambda(q_0, w)$. We write $w \geq u$, and say that u is a *transduct* of w, if there exists a FST A such that $w \geq_{\mathrm{A}} u$.

Note that the transducibility relation $\geq$ is a pre-order. It thus induces a partial order of 'degrees', the equivalence classes with respect to $\geq \cap \leq$. We denote equivalence using $\equiv$. It is not difficult to see that every word over a finite alphabet is equivalent to a word over the alphabet $\mathbf{2} = \{0, 1\}$. Thus every degree contains a representative from $\mathbf{2}^{\mathbb{N}}$. For the study of transducer degrees it suffices therefore to consider words over the latter alphabet.

Definition 4. Define the equivalence relation $\equiv \, = \, (\geq \cap \leq)$. The *(transducer) degree* $w^{\equiv}$ of an infinite word w is the equivalence class of w with respect to $\equiv$, that is, $w^{\equiv} = \{u \in \mathbf{2}^{\mathbb{N}} \mid w \equiv u\}$. We write $\mathbf{2}^{\mathbb{N}}/_{\equiv}$ to denote the set of degrees $\{w^{\equiv} \mid w \in \mathbf{2}^{\mathbb{N}}\}$.

The *transducer degrees* form the partial order $\langle \mathbf{2}^{\mathbb{N}}/_{\equiv}, \geq \rangle$, induced by the pre-order $\geq$ on $\mathbf{2}^{\mathbb{N}}$, that is, for words $w, u \in \mathbf{2}^{\mathbb{N}}$ we have $w^{\equiv} \geq u^{\equiv} \iff w \geq u$.

The *bottom degree* $\mathbf{0}$ is the least degree of the hierarchy, that is, the unique degree $\mathfrak{a} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ such that $\mathfrak{a} \leq \mathfrak{b}$ for every $\mathfrak{b} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$; it consists of the ultimately periodic words, that is, words of the form $uvvv\cdots$ for finite words u, v where $v \neq \varepsilon$.

Definition 5. An *atom* is a minimal non-bottom degree, that is, a degree $\mathfrak{a} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ such that $\mathbf{0} < \mathfrak{a}$ and there exists no $\mathfrak{b} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ with $\mathbf{0} < \mathfrak{b} < \mathfrak{a}$.

Although FSTs are very simple and elegant devices, we hardly understand their power for transforming words [10]. No methods are available to answer simple questions such as:

- Q4 Consider the period-doubling sequence P and drop every third element, resulting in $w = 01\_0\,0\_01\ \_10\_\ 01\_0\,0\_00\cdots$. Obviously we have $\mathsf{P} \geq w$. Do we have $w \geq \mathsf{P}$?
- Q5 Are the degrees of Thue–Morse T and Mephisto Waltz W incomparable?

4. Spiralling Words

We now consider *spiralling words* over the alphabet $\mathbf{2} = \{0, 1\}$ for which the distance of consecutive 1's in the word grows to infinity. We additionally require that the sequence of distances between consecutive 1's is ultimately periodic modulo every natural number. The class of spiralling words permits a characterisation of their transducts in terms of weighted products.

For a function $f : \mathbb{N} \to \mathbb{N}$, we define $\langle f \rangle \in \mathbf{2}^{\mathbb{N}}$ by

$$\langle f \rangle = \textstyle\prod_{i=0}^{\infty} 10^{f(i)} = 10^{f(0)}\,10^{f(1)}\,10^{f(2)}\,\cdots\ .$$

We write $\langle f(n) \rangle$ as shorthand for $\langle n \mapsto f(n) \rangle$.

Example 6. *As an example of a transduction between sporadic words, to get a feeling of what finite-state transducers can do on such words, consider $\langle n^3 \rangle \geq_A \langle (2n)^3 + (2n+1)^3 \rangle$. Here the transducer A removes the 1 between the appropriate consecutive blocks of 0's, as in: $1\,10\,10^8\,10^{27}\,10^{64}\,10^{125}\ 1\cdots \geq 10\,10^{(8+27)}\,10^{(64+125)}\,1\cdots$. It is easy to determine the two-state transducer A that removes the 1's at the right places.*

Definition 7. A function $f : \mathbb{N} \to \mathbb{N}$ is called *spiralling* if

(i) $\lim_{n\to\infty} f(n) = \infty$, and
(ii) for every $m \geq 1$, the function $n \mapsto f(n) \bmod m$ is ultimately periodic.

A word $\langle f \rangle$ is called *spiralling* whenever f is spiralling.

For example, $\langle p(n) \rangle$ is spiralling for every polynomial $p(n)$ with natural numbers as coefficients.[b] Spiralling functions are called 'cyclically ultimately periodic' in the literature [3]. For a tuple $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_m \rangle$, we define

- the *length* $|\vec{\alpha}| = m + 1$, and
- its *rotation* by $\vec{\alpha}' = \langle \alpha_1, \ldots, \alpha_m, \alpha_0 \rangle$.

[b]The identity function and constants functions are spiralling. Moreover, the class of spiralling functions is closed under addition and multiplication. From this it follows that polynomials with natural numbers as coefficients are spiralling.

Let A be a set and $f : \mathbb{N} \to A$ a function. We write $\mathcal{S}^k(f)$ for the *k-th shift* of f defined by $\mathcal{S}^k(f)(n) = f(n+k)$.

We use 'weights' to represent linear functions.

Definition 8. A *weight* $\vec{\alpha}$ is a tuple $\langle a_0, \ldots, a_{k-1}, b\rangle \in \mathbb{Q}^{k+1}$ of rational numbers such that $k \in \mathbb{N}$ and $a_0, \ldots, a_{k-1} \geq 0$. The weight $\vec{\alpha}$ is called

- *non-constant* if $a_i \neq 0$ for some $i < k$, else *constant*,
- *strongly non-constant* if $a_i, a_j \neq 0$ for some $i < j < k$.

Now let us also consider a tuple of tuples. For a tuple $\vec{\alpha} = \langle \vec{\alpha_0}, \ldots, \vec{\alpha_{m-1}}\rangle$ of weights we define $||\vec{\alpha}|| = \sum_{i=0}^{m-1}(\,|\vec{\alpha_i}| - 1\,)$.

Definition 9. Let $f : \mathbb{N} \to \mathbb{Q}$ be a function. For a weight $\vec{\alpha} = \langle a_0, \ldots, a_{k-1}, b\rangle$ we define $\vec{\alpha} \cdot f \in \mathbb{Q}$ by $\vec{\alpha} \cdot f \ = \ a_0 f(0) + a_1 f(1) + \cdots + a_{k-1} f(k-1) + b$. For a tuple of weights $\vec{\alpha} = \langle \vec{\alpha_0}, \vec{\alpha_1}, \ldots, \vec{\alpha_{m-1}}\rangle$, we define the *weighted product* $\vec{\alpha} \otimes f : \mathbb{N} \to \mathbb{Q}$ by induction on n:

$$
\begin{aligned}
(\vec{\alpha} \otimes f)(0) &= \vec{\alpha_0} \cdot f \\
(\vec{\alpha} \otimes f)(n+1) &= (\vec{\alpha}' \otimes \mathcal{S}^{|\vec{\alpha_0}|-1}(f))(n) \qquad\qquad (n \in \mathbb{N})
\end{aligned}
$$

We say that $\vec{\alpha} \otimes f$ is a *natural weighted product* if $(\vec{\alpha} \otimes f)(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Weighted products are easiest understood by examples.

Example 10. *Let $f(n) = n^2$ be a function and $\vec{\alpha} = \langle \vec{\alpha_0}, \vec{\alpha_1}\rangle$ a tuple of weights with $\vec{\alpha_0} = \langle 1, 2, 3, 4\rangle$, $\vec{\alpha_1} = \langle 0, 1, 1\rangle$. Then the weighted product $\vec{\alpha} \otimes f$ can be visualised as*

$$
\begin{array}{ccccccccccc}
f & 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81\cdots \\
 & \times 1 & \times 2 & \times 3\ {+4} & \times 0 & \times 1\ {+1} & \times 1 & \times 2 & \times 3\ {+4} & \times 0 & \times 1\ {+1} \\
\vec{\alpha} \otimes f & & 18 & & & 17 & & 248 & & 82 & \cdots
\end{array}
$$

Intuitively, the weight $\vec{\alpha_0} = \langle 1, 2, 3, 4\rangle$ means that 3 consecutive entries are added while being multiplied by 1, 2 and 3, respectively, and 4 is added to the result.

We introduce a few operations on weights. We define scalar multiplication of weights in the obvious way. We also introduce a multiplication $\odot$ that affects only the last entry of weights (the constant term).

Definition 11. Let $c \in \mathbb{Q}_{\geq 0}$, $\vec{\alpha} = \langle a_0, \ldots, a_{\ell-1}, b\rangle$ a weight, $\vec{\beta} = \langle \vec{\beta_0}, \ldots, \vec{\beta_{m-1}}\rangle$ a tuple of weights. We define

$$
\begin{aligned}
c\vec{\alpha} &= \langle ca_0, \ldots, ca_{\ell-1}, cb\rangle & \qquad \vec{\alpha} \odot c &= \langle a_0, \ldots, a_{\ell-1}, bc\rangle \\
c\vec{\beta} &= \langle c\vec{\beta_0}, \ldots, c\vec{\beta_{m-1}}\rangle & \qquad \vec{\beta} \odot c &= \langle \vec{\beta_0} \odot c, \ldots, \vec{\beta_{m-1}} \odot c\rangle
\end{aligned}
$$

The next lemma follows directly from the definitions.

Lemma 12. *Let $c \in \mathbb{Q}_{\geq 0}$, $\vec{\alpha}$ a tuple of weights, and $f : \mathbb{N} \to \mathbb{Q}$ a function. Then $c(\vec{\alpha} \otimes f) = (c\vec{\alpha}) \otimes f = (\vec{\alpha} \odot c) \otimes (cf)$.* $\square$

It is straightforward to define a *composition* of tuples of weights such that $\vec{\beta} \otimes (\vec{\alpha} \otimes f) = (\vec{\beta} \otimes \vec{\alpha}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$. Note that $\vec{\alpha} \otimes f$ is already defined. For the precise definition of $\vec{\beta} \otimes \vec{\alpha}$, we refer to [9]. We will employ the following two properties of composition.

Lemma 13. *Let $\vec{\alpha}, \vec{\beta}$ be tuples of weights. Then we have that $\vec{\beta} \otimes (\vec{\alpha} \otimes f) = (\vec{\beta} \otimes \vec{\alpha}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$.* $\square$

Lemma 14. *Let $\vec{\alpha}$ be tuple of weights, and $\vec{\beta}$ a tuple of strongly non-constant weights. Then $\vec{\alpha} \otimes \vec{\beta}$ is of the form $\langle \gamma_0, \ldots, \gamma_{k-1} \rangle$ such that for every $i \in \mathbb{N}_{<k}$, the weight γ_i is either constant or strongly non-constant.* $\square$

We need a few results on weighted products from [5]. The following lemma states that every natural weighted product (see Definition 9) can be realised by a FST.

Lemma 15 ([5]) *Let $f : \mathbb{N} \to \mathbb{N}$, and $\vec{\alpha}$ a tuple of weights. If $\vec{\alpha} \otimes f$ is a natural weighted product (i.e., $\forall n \in \mathbb{N}.\ (\vec{\alpha} \otimes f)(n) \in \mathbb{N}$), then $\langle f \rangle \geq \langle \vec{\alpha} \otimes f \rangle$.* $\square$

For the proof of Theorem 22, below, we use the following auxiliary lemma. The lemma gives a detailed structural analysis, elaborated and explained in [5], of the transducts of a spiralling word $\langle f \rangle$.

Lemma 16 ([5]) *Let $f : \mathbb{N} \to \mathbb{N}$ be a spiralling function, and let $\sigma \in \mathbf{2}^{\mathbb{N}}$ be such that $\langle f \rangle \geq \sigma$ and $\sigma \notin \mathbf{0}$. Then there exist $n_0, m \in \mathbb{N}$, a word $w \in \mathbf{2}^*$, a tuple of weights $\vec{\alpha}$, and tuples of finite words $\vec{p}$ and $\vec{c}$ with $|\vec{\alpha}| = |\vec{p}| = |\vec{c}| = m > 0$ such that $\sigma = w \cdot \prod_{i=0}^{\infty} \prod_{j=0}^{m-1} p_j\, c_j^{\varphi(i,j)}$ where $\varphi(i,j) = (\vec{\alpha} \otimes \mathcal{S}^{n_0}(f))(mi + j)$, and*

(i) $c_j^{\omega} \neq p_{j+1} c_{j+1}^{\omega}$ for every j with $0 \leq j < m-1$, and $c_{m-1}^{\omega} \neq p_0 c_0^{\omega}$, and
(ii) $c_j \neq \varepsilon$, and α_j is non-constant, for all $j \in \mathbb{N}_{<m}$. $\square$

Example 17. *We continue Example 10. We have $\vec{\alpha} = \langle \vec{\alpha_0}, \vec{\alpha_1} \rangle$. Accordingly, we have prefixes $p_0, p_1 \in \mathbf{2}^*$ and cycles $c_0, c_1 \in \mathbf{2}^*$. Then the transduct σ in Lemma 16, defined by the double product, can be derived as follows:*

$$\begin{array}{cccccccccccc}
f & 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & \cdots \\
 & \multicolumn{3}{c}{\vec{\alpha_0}} & \multicolumn{2}{c}{\vec{\alpha_1}} & \multicolumn{3}{c}{\vec{\alpha_0}} & \multicolumn{2}{c}{\vec{\alpha_1}} & \\
\vec{\alpha} \otimes f & \multicolumn{3}{c}{18} & \multicolumn{2}{c}{17} & \multicolumn{3}{c}{248} & \multicolumn{2}{c}{82} & \cdots \\
\sigma = w \cdot & \multicolumn{3}{c}{p_0\, c_0^{18} \cdot} & \multicolumn{2}{c}{p_1\, c_1^{17} \cdot} & \multicolumn{3}{c}{p_0\, c_0^{248} \cdot} & \multicolumn{2}{c}{p_1\, c_1^{82}} & \cdots
\end{array}$$

The infinite word σ is the infinite concatenation of w followed by alternating $p_0 c_0^{e_0}$ and $p_1 c_1^{e_1}$, where the exponents e_0 and e_1 are the result of applying weights $\vec{\alpha_0}$ and $\vec{\alpha_1}$, respectively.

We characterise the transducts of spiralling words up to equivalence ($\equiv$).

Theorem 18 ([5]) *Let $f : \mathbb{N} \to \mathbb{N}$ be spiralling, and $\sigma \in \mathbf{2}^{\mathbb{N}}$. Then $\langle f \rangle \geq \sigma$ if and only if $\sigma \equiv \langle \vec{\alpha} \otimes \mathcal{S}^{n_0}(f) \rangle$ for some $n_0 \in \mathbb{N}$, and a tuple of weights $\vec{\alpha}$.*

Roughly speaking, polynomials of order k are closed under transduction.

Let $p(n)$ be a polynomial of order k with non-negative integer coefficients, and let $\sigma \notin \mathbf{0}$ with $\langle p(n)\rangle \geq \sigma$. Then $\sigma \geq \langle q(n)\rangle$ for some polynomial $q(n)$ of order k with non-negative integer coefficients.

5. The Degree of $\langle n^k\rangle$ is Not an Atom for $k \geq 3$

We show that the degree of $\langle n^k\rangle$ is not an atom for $k \geq 3$. For this purpose, we prove a strengthening of Theorem 18, a lemma on weighted products of strongly non-constant weights, and we employ the power mean inequality [11].

Definition 20. For $p \in \mathbb{R}$, the *weighted power mean* $M_p(\vec{x})$ of $\vec{x} = \langle x_1, x_2, \dots, x_n\rangle \in \mathbb{R}^n_{>0}$ with respect to $\vec{w} = \langle w_1, w_2, \dots, w_n\rangle \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n w_i = 1$ is

$$M_{\vec{w},0}(\vec{x}) = \prod_{i=1}^n x_i^{w_i} \qquad\qquad M_{\vec{w},p}(\vec{x}) = (\sum_{i=1}^n w_i x_i^p)^{1/p}\,.$$

Proposition 21 (Power mean inequality) *For all $p, q \in \mathbb{R}$, $\vec{x}, \vec{w} \in \mathbb{R}^n_{>0}$:*

$$p < q \implies M_{\vec{w},p}(\vec{x}) \leq M_{\vec{w},q}(\vec{x})$$
$$(p = q \vee x_1 = x_2 = \cdots = x_n) \iff M_{\vec{w},p}(\vec{x}) = M_{\vec{w},q}(\vec{x})\,.$$

Theorem 18 characterises transducts of spiralling sequences only up to equivalence. This makes it difficult to employ the theorem for proving non-transducibility. We improve the characterisation for the case of spiralling transducts as follows.

Theorem 22. *Let $f, g : \mathbb{N} \to \mathbb{N}$ be spiralling functions. Then $\langle g\rangle \geq \langle f\rangle$ if and only if some shift of f is a weighted product of a shift of g, that is:*

$$\mathcal{S}^{n_0}(f) = \vec{\alpha} \otimes \mathcal{S}^{m_0}(g)$$

for some $n_0, m_0 \in \mathbb{N}$ and a tuple of weights $\vec{\alpha}$.

Proof. For the direction '$\Leftarrow$', assume that $\mathcal{S}^{n_0}(f) = \vec{\alpha} \otimes \mathcal{S}^{m_0}(g)$. Then we have $\langle g\rangle \equiv \langle \mathcal{S}^{m_0}(g)\rangle \geq \langle \vec{\alpha} \otimes \mathcal{S}^{m_0}(g)\rangle = \langle \mathcal{S}^{n_0}(f)\rangle \equiv \langle f\rangle$ by invariance under shifts and by Lemma 15.

For the direction '$\Rightarrow$', assume that $\langle g\rangle \geq \langle f\rangle$. Then by Lemma 16 there exist $m_0, m \in \mathbb{N}$, $w \in \mathbf{2}^*$, $\vec{\alpha}$, $\vec{p}$ and $\vec{c}$ with $|\vec{\alpha}| = |\vec{p}| = |\vec{c}| = m > 0$ such that:

$$\langle f\rangle = w \cdot \prod_{i=0}^{\infty} \prod_{j=0}^{m-1} p_j\, c_j^{\varphi(i,j)} \tag{1}$$

where $\varphi(i,j) = (\vec{\alpha} \otimes \mathcal{S}^{m_0}(g))(mi+j)$ such that the conditions (i) and (ii) of Lemma 16 are fulfilled.

Note that, as $\lim_{n\to\infty} f(n) = \infty$, the distance of ones in the sequence $\langle g\rangle$ tends to infinity. For every $j \in \mathbb{N}_{<m}$, the word p_j occurs infinitely often in $\langle f\rangle$ by (1), and hence p_j can contain at most one occurrence of the symbol 1.

By condition (ii), we have for every $j \in \mathbb{N}_{<m}$ that $c_j \neq \varepsilon$, and the weight $\vec{\alpha_j}$ is not constant. As $\lim_{n\to\infty} g(n) = \infty$, it follows that c_j^2 appears infinitely often in $\langle f\rangle$ by (1). Hence c_j consists only of 0's, that is, $c_j \in \{0\}^+$ for every $j \in \mathbb{N}_{<m}$.

By condition (i) we never have $c_j^\omega = p_{j+1} c_{j+1}^\omega$ for $j \in \mathbb{N}_{<m}$ (where addition is modulo m). As $c_j^\omega = 0^\omega$ and $p_{j+1} 0^\omega = p_{j+1} c_{j+1}^\omega$, we obtain that p_{j+1} must contain a 1. Hence, for every $k \in \mathbb{N}_{<m}$, the word p_j contains precisely one 1.

Finally, we apply the following transformations to ensure $p_j = 1$ and $c_j = 0$ for every $j \in \mathbb{N}_{<m}$:

(i) For every $j \in \mathbb{N}_{<m}$ such that $c_j = 0^h$ for some $h > 1$, we set $c_j = 0$ and replace the weight $\vec{\alpha_j}$ in $\vec{\alpha}$ by $h\vec{\alpha_j}$.

(ii) For every $j \in \mathbb{N}_{<m}$ such that $p_j = 0^h 1 0^\ell$ for some $h \geq 1$ or $\ell \geq 1$, we set $p_j = 1$ and replace the weight $\vec{\alpha_j}$ in $\vec{\alpha}$ by $(\vec{\alpha_j} + \ell)$ and the weight $\vec{\alpha_{j-1}}$ by $(\vec{\alpha_{j-1}} + h)$. Here, for a weight $\vec{\gamma} = \langle x_0, \ldots, x_{\ell-1}, y \rangle$ and $z \in \mathbb{Q}$, we write $\vec{\gamma} + z$ for the weight $\langle x_0, \ldots, x_{\ell-1}, y + z \rangle$. If $j = 0$, we moreover append 0^h to the word w.

Both transformations preserve equation (1); the double product remains the same.

Thus we now have $p_j = 1$ and $c_j = 0$ for every $j \in \mathbb{N}_{<m}$. It follows from (1) that $\langle f \rangle = w \langle \vec{\alpha} \otimes \mathcal{S}^{m_0}(g) \rangle$. Hence $\mathcal{S}^{n_0}(f) = \vec{\alpha} \otimes \mathcal{S}^{m_0}(g)$ for some $n_0 \in \mathbb{N}$. □

Theorem 22 strengthens Theorem 18 in the sense that the characterisation uses equality (= and shifts) instead of equivalence ($\equiv$). But Theorem 22 only characterises spiralling transducts whereas Theorem 18 characterises all transducts. However, next we will employ the gained precision to show that certain spiralling transducts of $\langle n^k \rangle$ cannot be transduced back to $\langle n^k \rangle$, and conclude that $\langle n^k \rangle$ is not an atom for $k \geq 3$.

Theorem 23. *For $k \geq 3$, the degree of $\langle n^k \rangle$ is not an atom.*

Proof. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = n^k$. We have $\langle f \rangle \geq \langle g \rangle$ where $g : \mathbb{N} \to \mathbb{N}$ is defined by $g(n) = (2n)^k + (2n+1)^k$; cf. Example 6. Assume that we had $\langle g \rangle \geq \langle f \rangle$. Then, by Theorem 22 we have $\mathcal{S}^{n_0}(f) = \vec{\alpha} \otimes \mathcal{S}^{m_0}(g)$ for some $n_0, m_0 \in \mathbb{N}$ and a tuple of weights $\vec{\alpha}$. Note that $g = \langle \langle 1, 1, 0 \rangle \rangle \otimes f$ and

$$\begin{aligned} \mathcal{S}^{n_0}(f) &= \vec{\alpha} \otimes \mathcal{S}^{m_0}(\langle \langle 1, 1, 0 \rangle \rangle \otimes f) \\ &= \vec{\alpha} \otimes (\langle \langle 1, 1, 0 \rangle \rangle \otimes \mathcal{S}^{2m_0}(f)) = \vec{\beta} \otimes \mathcal{S}^{2m_0}(f) \end{aligned}$$

where $\vec{\beta} = \vec{\alpha} \otimes \langle \langle 1, 1, 0 \rangle \rangle$. By Lemma 14 every weight in $\vec{\beta}$ is either constant or strongly non-constant. As $\mathcal{S}^{n_0}(f)$ is strictly increasing (and hence contains no constant subsequence), each weight in $\vec{\beta}$ must be strongly non-constant.

Let $\vec{\beta} = \langle \vec{\beta_0}, \ldots, \vec{\beta_{\ell-1}} \rangle$. For every $n \in \mathbb{N}$ we have

$$\mathcal{S}^{n_0}(f)(\ell n) = (\vec{\beta} \otimes \mathcal{S}^{2m_0}(f))(\ell n) = \vec{\beta_0} \cdot \mathcal{S}^{2m_0 + ||\vec{\beta}|| \cdot n}(f) \; . \tag{2}$$

Then we have

$$\begin{aligned} \mathcal{S}^{n_0}(f)(\ell n) &= (n_0 + \ell n)^k = \textstyle\sum_{i=0}^{k} \binom{k}{i} n_0^i \ell^{k-i} n^{k-i} \\ &= \ell^k n^k + k n_0 \ell^{k-1} n^{k-1} + \cdots + k n_0^{k-1} \ell n + n_0^k \; . \end{aligned} \tag{3}$$

Let $\vec{\beta_0} = \langle a_0, a_1, \ldots, a_{h-1}, b\rangle$. We define $c_i = a_i||\vec{\beta}||^k$ and $d_i = (2m_0 + i)/||\vec{\beta}||$. We obtain

$$\begin{aligned}
\vec{\beta_0} \cdot \mathcal{S}^{2m_0+||\vec{\beta}||\cdot n}(f) &= b + \textstyle\sum_{i=0}^{h-1} a_i f(2m_0 + ||\vec{\beta}|| \cdot n + i) \\
&= b + \textstyle\sum_{i=0}^{h-1} a_i f(||\vec{\beta}||(n + \frac{2m_0+i}{||\vec{\beta}||})) \\
&= b + \textstyle\sum_{i=0}^{h-1} a_i ||\vec{\beta}||^k (n + d_i)^k = b + \sum_{i=0}^{h-1} c_i (n + d_i)^k \\
&= b + \textstyle\sum_{i=0}^{h-1} c_i (n^k + k d_i n^{k-1} + \cdots + k d_i^{k-1} n + d_i^k) \,. \qquad (4)
\end{aligned}$$

Recall equation (2). Comparing the coefficients of n^k, n^{k-1} and n in (3) and (4) we obtain

$$\begin{aligned}
\ell^k &= \sum_{i=0}^{h-1} c_i & k n_0 \ell^{k-1} &= \sum_{i=0}^{h-1} c_i k d_i & k n_0^{k-1} \ell &= \sum_{i=0}^{h-1} c_i k d_i^{k-1} \text{, and hence} \\
1 &= \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} & \frac{n_0}{\ell} &= \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} d_i & \frac{n_0^{k-1}}{\ell^{k-1}} &= \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} d_i^{k-1} \,,
\end{aligned}$$

contradicting the weighted power means inequality (Proposition 21). Clearly all d_i are distinct, and, as a consequence of $\vec{\beta_0}$ being strongly non-constant, there are at least two $i \in \mathbb{N}_{<h}$ for which $c_i \neq 0$. Thus our assumption $\langle g\rangle \geq \langle f\rangle$ is wrong. Hence the degree of $\langle n^k\rangle$ is not an atom. □

6. Atoms of Every Polynomial Order

The previous section stated that $\langle n^k\rangle$ is not an atom for $k \geq 3$. Now we show that for every $k \in \mathbb{N}$ there exists a polynomial $p(n)$ of order k such that the degree of the word $\langle p(n)\rangle$ is an atom. Hence there are at least $\aleph_0$ atoms in the transducer degrees.

As we have seen in the proof of Theorem 23, whenever $k \geq 3$, we have that $\langle n^k\rangle \geq \langle g(n)\rangle$, but not $\langle g(n)\rangle \geq \langle n^k\rangle$ for $g(n) = (2n)^k + (2n+1)^k$. Thus there exist polynomials $p(n)$ of order k for which $\langle p(n)\rangle$ cannot be transduced to $\langle n^k\rangle$. However, the key observation underlying the construction in this section is the following: Although we may not be able to reach $\langle n^k\rangle$ from $\langle p(n)\rangle$, we can get arbitrarily close (Lemma 26, below). This enables us to employ the concept of *continuity.*

In order to have continuous functions over the space of polynomials to allow limit constructions, we now permit rational coefficients. For $k \in \mathbb{N}$, let $\mathfrak{Q}_k$ be the set of polynomials of order k with non-negative rational coefficients. We also use polynomials in $\mathfrak{Q}_k$ to denote spiralling sequences. However, we need to give meaning to $\langle q(n)\rangle$ for the case that the block sizes $q(n)$ are not natural numbers. For this purpose, we make use of the fact that the degree of a word $\langle f(n)\rangle$ is invariant under multiplication of the block sizes by a constant, as is easy to see. More precisely, for $f : \mathbb{N} \to \mathbb{N}$, we have $\langle f(n)\rangle \equiv \langle d \cdot f(n)\rangle$ for every $d \in \mathbb{N}$ with $d \geq 1$. So to give meaning to $\langle q(n)\rangle$, we multiply the polynomial by the least natural number $d > 0$ such that $d \cdot q(n)$ is a natural number for every $n \in \mathbb{N}$.

Definition 24. We call a function $f : \mathbb{N} \to \mathbb{Q}$ *naturalisable* if there exists a natural number $d \geq 1$ such that for all $n \in \mathbb{N}$ we have $(d \cdot f(n)) \in \mathbb{N}$.

For naturalisable $f : \mathbb{N} \to \mathbb{Q}$ we define $\langle f \rangle = \langle d \cdot f \rangle$ where $d \in \mathbb{N}$ is the least number such that $d \geq 1$ where for all $n \in \mathbb{N}$ we have $(d \cdot f(n)) \in \mathbb{N}$. (Note that, for $f : \mathbb{N} \to \mathbb{N}$, $\langle f(n) \rangle$ has been defined in Section 4.)

Observe that every $q(n) \in \mathfrak{Q}_k$ is naturalisable (multiply by the least common denominator of the coefficients). Also, naturalisable functions are preserved under weighted products.

Lemma 15 generalises as follows. We no longer need to require that the weighted product is natural. All weighted products of naturalisable functions can be realised by finite-state transducers.

Lemma 25. *Let $f : \mathbb{N} \to \mathbb{Q}$ be naturalisable, and $\vec{\alpha}$ a tuple of weights. Then $\vec{\alpha} \otimes f$ is naturalisable and $\langle f \rangle \geq \langle \vec{\alpha} \otimes f \rangle$.*

Proof. Let $\vec{\alpha} = \langle \vec{\alpha_0}, \ldots, \vec{\alpha_{m-1}} \rangle$ for some $m \geq 1$. Let $c \in \mathbb{N}$ with $c \geq 1$ be minimal such that all entries of $c\vec{\alpha}$ are natural numbers. Let $d \in \mathbb{N}$ with $d \geq 1$ be the least natural number such that $\forall n \in \mathbb{N}\ (d \cdot f(n)) \in \mathbb{N}$.

Then we obtain $((dc\vec{\alpha}) \otimes f)(n) \in \mathbb{N}$ for ever $n \in \mathbb{N}$. By the definition of weighted products it follows immediately that $(dc\vec{\alpha}) \otimes f = dc(\vec{\alpha} \otimes f)$, and hence $\vec{\alpha} \otimes f$ is naturalisable. Let $e \in \mathbb{N}$ with $e \geq 1$ be the least natural number such that $\forall n \in \mathbb{N}\ (e \cdot (\vec{\alpha} \otimes f)(n)) \in \mathbb{N}$.

We have the following transduction

$$
\begin{aligned}
\langle f \rangle &= \langle df \rangle && \text{by Definition 24} \\
&\geq \langle ((c\vec{\alpha}) \odot d) \otimes (df) \rangle && \text{by Lemma 15} \\
&= \langle (dc\vec{\alpha}) \otimes f \rangle = \langle dc(\vec{\alpha} \otimes f) \rangle && \text{by Lemma 12} \\
&\geq \langle \langle \langle \frac{e}{dc}, 0 \rangle \rangle \otimes (dc(\vec{\alpha} \otimes f)) \rangle && \text{by Lemma 15} \\
&= \langle e(\vec{\alpha} \otimes f) \rangle = \langle \vec{\alpha} \otimes f \rangle && \text{by Definition 24}
\end{aligned}
$$

This concludes the proof. □

The following lemma states that every word $\langle q(n) \rangle$, for a polynomial $q(n) \in \mathfrak{Q}_k$ of order k, can be transduced arbitrarily close to (but perhaps not equal to) $\langle n^k \rangle$.

Lemma 26. *Let $k \geq 1$ and let $q(n) \in \mathfrak{Q}_k$ be a polynomial of order k. For every $\varepsilon > 0$ we have $\langle q(n) \rangle \geq \langle n^k + b_{k-1}n^{k-1} + \cdots + b_1 n \rangle$ for some rational coefficients $0 \leq b_{k-1}, \ldots, b_1 < \varepsilon$.*

Proof. Let $q(n) = a_k n^k + a_{k-1}n^{k-1} + \cdots + a_1 n + a_0$, and let $\varepsilon > 0$ be arbitrary. Then for every $d \in \mathbb{N}$, we have

$$\langle q(n) \rangle \geq \langle q(dn) \rangle \geq \langle \frac{q(dn)}{a_k d^k} \rangle = \langle n^k + \frac{a_{k-1}}{a_k d} n^{k-1} + \cdots + \frac{a_1}{a_k d^{k-1}} n^1 + \frac{a_0}{a_k d^k} \rangle$$

$$\geq \langle n^k + \frac{a_{k-1}}{a_k d} n^{k-1} + \ldots + \frac{a_1}{a_k d^{k-1}} n^1 \rangle$$

The first transduction selects a subsequence of the blocks. The second transduction is a division of the size of each block (application of Lemma 25 with the weight $\langle\langle 1/a_k d^k, 0\rangle\rangle$). The last transduction amounts to removing a constant number of zeros from each block (application of Lemma 25 with the weight $\langle\langle 1, -a_0/(a_k d^k)\rangle\rangle$). The last polynomial in the transduction is of the desired form if $d \in \mathbb{N}$ is chosen large enough. □

For polynomials $p(n) \in \mathfrak{Q}_k$, we want to express weighted products $\langle\vec{\alpha}\rangle \otimes p$ in terms of matrix products, as follows.

Definition 27. For weights $\vec{\alpha} = \langle a_0, \ldots, a_{k-1}, b\rangle$ we define a column vector

$$U(\vec{\alpha}) = (a_0, \ldots, a_{k-1})^T .$$

Definition 28. If $p(n) = \sum_{i=0}^{k} c_i n^i$ is a polynomial of order k, we define a column vector $V(p(n)) = (c_1, \ldots, c_k)^T$ and a square matrix

$$M(p(n)) = (V(p(kn+0)),\ \ldots,\ V(p(kn+k-1))) .$$

We also write $V(p)$ short for $V(p(n))$ and $M(p)$ for $M(p(n))$.

We have omitted the constant term c_0 from the definition of $V(p)$. Because for every $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{N}$ we have $\langle f(n)\rangle \equiv \langle f(n)+c\rangle$. These words are of the same degree because a FST can add (or remove) a constant number of symbols 0 to (from) every block of 0's. Similarly, b was omitted from the definition of $U(\vec{\alpha})$.

Example 29. *Consider the polynomial n^3:*

$$V(n^3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad M(n^3) = \begin{pmatrix} 0 & 9 & 36 \\ 0 & 27 & 54 \\ 27 & 27 & 27 \end{pmatrix}$$

where the column vectors of the matrix $M(n^3)$ are given by $V((3n)^3)$, $V((3n+1)^3)$ and $V((3n+2)^3)$.

Lemma 30. *Let $k \geq 1$. Let $\vec{\alpha} = \langle a_0, \ldots, a_{k-1}, b\rangle$ be a weight and $p(n) \in \mathfrak{Q}_k$. Then $M(p)\,U(\vec{\alpha}) = V(\langle\vec{\alpha}\rangle \otimes p)$.*

Proof. We calculate

$$\begin{aligned} M(p)\,U(\vec{\alpha}) &= \sum_{i=0}^{k-1} a_i V(p(kn+i)) = V\Big(\sum_{i=0}^{k-1} a_i p(kn+i)\Big) \\ &= V\Big(\sum_{i=0}^{k-1} a_i p(kn+i) + b\Big) = V(\langle\vec{\alpha}\rangle \otimes p) , \end{aligned}$$

which proves the lemma. □

Let us take a closer look at the matrix $M(n^k)$. The element in the ith row and jth column is $M_{i,j} = \binom{k}{i} k^i (j-1)^{k-i}$. Dividing the ith row by $\binom{k}{i} k^i$ for each i gives a Vandermonde-type matrix, which is invertible. Thus also $M(n^k)$ is invertible.

Lemma 31. *For $k \geq 1$, $M(n^k)$ is invertible.* □

Theorem 32. *Let $k \geq 1$. Let $a_0, \dots, a_{k-1}$ be positive rational numbers, $\vec{\alpha} = \langle a_0, \dots, a_{k-1}, 0 \rangle$, and*

$$p(n) = (\langle \vec{\alpha} \rangle \otimes n^k)(n) = \sum_{i=0}^{k-1} a_i (kn + i)^k.$$

Then $\langle q(n) \rangle \geq \langle p(n) \rangle$ for all $q(n) \in \mathfrak{Q}_k$. Moreover, the degree $\langle p(n) \rangle^{\equiv}$ is an atom. Note that the degree $\langle p(n) \rangle^{\equiv}$ is the infimum of all degrees of words $\langle q(n) \rangle$ with $q(n) \in \mathfrak{Q}_k$.

Proof. By Lemma 30, $M(n^k)\, U(\vec{\alpha}) = V(p)$. By Lemma 31, $M(n^k)$ is invertible and we can write $U(\vec{\alpha}) = M(n^k)^{-1} V(p)$. By Lemma 26, for every $\varepsilon > 0$ there exists $q_\varepsilon \in \mathfrak{Q}_k$ such that $\langle q(n) \rangle \geq \langle q_\varepsilon(n) \rangle$ and

$$q_\varepsilon(n) = n^k + b_{k-1} n^{k-1} + \cdots + b_1 n$$

with $0 \leq b_i \leq \varepsilon$ for every $i \in \{1, \dots, k-1\}$. We will show that if ε is small enough, then $\langle q_\varepsilon(n) \rangle \geq \langle p(n) \rangle$.

We have $\lim_{\varepsilon \to 0} M(q_\varepsilon) = M(n^k)$. As $\det(M(n^3)) \neq 0$ and the determinant function is continuous, also $\det(M(q_\varepsilon)) \neq 0$ for all sufficiently small ε. Then $M(q_\varepsilon)$ is invertible, and we define $U_\varepsilon = M(q_\varepsilon)^{-1} V(p)$. We would like to have $U_\varepsilon = U(\gamma)$ for some weight γ. This is not always possible, because some elements of U_ε might be negative. However, by the continuity of matrix inverse and product,

$$\lim_{\varepsilon \to 0} U_\varepsilon = \lim_{\varepsilon \to 0} (M(q_\varepsilon)^{-1} V(p)) = (\lim_{\varepsilon \to 0} M(q_\varepsilon))^{-1} V(p) = M(n^k)^{-1} V(p) = U(\vec{\alpha}).$$

Since every element of $U(\vec{\alpha})$ is positive, we can fix a small enough ε so that every element of U_ε is positive. Then we have $U_\varepsilon = U(\gamma)$ for some weight γ.

We have $M(q_\varepsilon)\, U(\gamma) = V(\langle \gamma \rangle \otimes q_\varepsilon)$ by Lemma 30, and $M(q_\varepsilon)\, U(\gamma) = V(p)$ by the definition of U_ε. As a consequence $(\langle \gamma \rangle \otimes q_\varepsilon)(n) = p(n) + c$ for some constant c. By Lemma 25, we obtain $\langle q_\varepsilon(n) \rangle \geq \langle p(n) \rangle$.

It remains to show that the degree $\langle p(n) \rangle^{\equiv}$ is an atom. Assume that $\langle p(n) \rangle \geq w$ and $w \notin \mathbf{0}$. By Proposition 19 we have $w \geq \langle q(n) \rangle$ for some $q(n) \in \mathfrak{Q}_k$. As shown above, $\langle q(n) \rangle \geq \langle p(n) \rangle$, thus $w \geq \langle p(n) \rangle$. Hence $\langle p(n) \rangle^{\equiv}$ is an atom. □

7. A Hereditary Uncomputable Degree

We show that for any countable set $\mathfrak{D}$ of transducer degrees that does not contain the bottom degree, there exists a degree $\mathfrak{z} \neq \mathbf{0}$ that such that $\mathfrak{z}{\downarrow}$ contains no degree from $\mathfrak{D}$. Here $\mathfrak{z}{\downarrow}$ is the *cone* of $\mathfrak{z}$, that is, the set of degrees below $\mathfrak{z}$:

$$\mathfrak{z}{\downarrow} \;=\; \{\, \mathfrak{a} \mid \mathfrak{z} \geq \mathfrak{a} \,\}\,.$$

To this end, we will prove the following theorem.

Theorem 33. *Let $\mathcal{S} \subseteq \mathbf{2}^{\mathbb{N}}$ be a countable set of words that contains no ultimately periodic words. Then there exists a word $w \in \mathbf{2}^{\mathbb{N}}$ that is not ultimately periodic and none of the transducts u of w, $w \geq u$, resides in $\mathcal{S}$.*

Before proving this theorem, we mention a few corollaries.

Corollary 34. *There exists an uncomputable word $\mathsf{U} \in \mathbf{2}^{\mathbb{N}}$ whose finite-state transducts are all uncomputable, ultimately periodic or finite.*

Proof. Follows from Theorem 33 with S the set of computable words that are not ultimately periodic. □

Theorem 33 and Corollary 34 have the following immediate implications for the hierarchy of transducer degrees.

Corollary 35. *Let $\mathfrak{D}$ be a countable set of transducer degrees not containing the bottom degree. Then there exists a degree $\mathfrak{z} \neq \mathbf{0}$ that has no degrees in $\mathfrak{D}$ below itself, that is, $\mathfrak{z}{\downarrow} \cap \mathfrak{D} = \varnothing$.*

The following result is somewhat reminiscent of the situation in Turing degrees where there exists a set of incomparable degrees of size continuum.

Corollary 36. *Let $\mathfrak{C}$ be a countable set of degrees with pairwise almost disjoint cones, that is, for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}$ with $\mathfrak{a} \neq \mathfrak{b}$, we have $\mathfrak{a}{\downarrow} \cap \mathfrak{b}{\downarrow} = \{\mathbf{0}\}$. Then $\mathfrak{C}$ can be extended to an uncountable set of degrees with pairwise almost disjoint cones.*

Proof. Let $\mathfrak{C}'$ be a maximal extension of $\mathfrak{C}$. If $\mathfrak{C}'$ was countable, then by Corollary 35 it could be extended by a disjoint cone: take $\mathfrak{D} = \{\mathfrak{b} \mid \mathfrak{a} \in \mathfrak{C}', \mathfrak{a} \geq \mathfrak{b}\} \setminus \{\mathbf{0}\}$. This contradicts maximality of $\mathfrak{C}'$. □

We do not know if 'uncountable' can be replaced by continuum in the corollary.

We call a degree *uncomputable* if it contains an uncomputable word. Note that degrees cannot contain both computable and uncomputable words since the set of computable words is closed under finite-state transduction.

Corollary 37. *There exists an uncomputable transducer degree $\mathsf{U}^{\equiv}$ that has only uncomputable degrees and the bottom degree below itself.*

For the proof of Theorem 33 we introduce a few auxiliary definitions and lemmas.

Definition 38. *Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite-state transducer, and $w \in \Sigma^*$ a word. Then A is* predetermined *by w if there exists $u \in \Gamma^{\mathbb{N}}$ such that for every $w' \in \Sigma^*$ it holds that $\lambda(q_0, ww') \sqsubseteq u$.*

When a transducer is predetermined by w, then it transduces words starting with w to ultimately periodic words (or finite words).

Lemma 39. *Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be predetermined by $w \in \Sigma^*$, and let $w' \in \Sigma^{\mathbb{N}}$. If the word $\lambda(q_0, ww')$ is infinite, then it is ultimately periodic.*

Proof. Let $u \in \Gamma^{\mathbb{N}}$ such that

$$\forall w' \in \Sigma^* \ \lambda(q_0, ww') \sqsubseteq u \tag{5}$$

Let $w' \in \Sigma^{\mathbb{N}}$ such that $\lambda(q_0, ww')$ is infinite. Note that from (5) it follows that $\lambda(q_0, ww') = u$. Let $q = \delta(q_0, w)$. Then $\lambda(q_0, ww') = \lambda(q_0, w)\lambda(q, w')$. By the infinitary pigeonhole principle, there exists some state $q' \in Q$ that is visited infinitely often when the automaton reads w' starting in state q. Consequently there are non-empty words $w_0, w_1, \ldots \in \Sigma^+$ such that $w' = w_0 w_1 w_2 \cdots$, and for every $n \in \mathbb{N}$ we have that $\delta(q, w_0 w_1 \cdots w_n) = q'$. As $\lambda(q_0, ww')$ and hence $\lambda(q, w')$ is infinite, there exists some $i > 0$ such that $\lambda(q', w_i) \neq \varepsilon$. Define $v = \lambda(q, w_0 w_1 \cdots w_{i-1} w_i w_i w_i w_i \cdots)$, then it follows that v is infinite. Moreover v is ultimately periodic as it is the transduct of an ultimately periodic sequence. We obtain $\lambda(q_0, w)v = u$ from (5), and thus $u = \lambda(q_0, ww')$ is ultimately periodic. $\square$

We are now ready to prove Theorem 33.

Proof of Theorem 33. Let $\mathcal{S} \subseteq \mathbf{2}^{\mathbb{N}}$ be a countable set of words that contains no ultimately periodic words. Let $\mathcal{A}$ be the set of all finite-state transducers over the alphabet $\mathbf{2}$. Note that $\mathcal{A} \times \mathcal{S}$ is countable and let $(A_0, s_0), (A_1, s_1), \ldots$ be an enumeration of this set. For $i = 0, 1, 2, \ldots$ we define words $w_i \in \mathbf{2}^+$ as follows. Let $v_i = w_0 \cdots w_{i-1}$. We stipulate that $v_0 = \varepsilon$. Let $A_i = \langle \mathbf{2}, \mathbf{2}, Q, q_0, \delta, \lambda \rangle$. If A_i is predetermined by v_i, then the choice of w_i is arbitrary; say $w_i = 0$. Otherwise, there exist words $x, y \in \Sigma^+$ such that neither $x' \sqsubseteq y'$ nor $y' \sqsubseteq x'$, where $x' = \lambda(q_0, v_i x)$ and $y' = \lambda(q_0, v_i y)$. Then there exists an index $j < \min\{|x'|, |y'|\}$ such that $x'(j) \neq y'(j)$. Define $w_i = y$ if $x'(j) = s_i(j)$, and $w_i = x$, otherwise. This choice guarantees that

$$\lambda(q_0, w_0 \cdots w_i) \not\sqsubseteq s_i \tag{6}$$

Let $w = w_0 w_1 w_2 \cdots$. Assume that there exists $u \in \mathcal{S}$ with $w \geq u$. Then there exist a finite-state transducer $A \in \mathcal{A}$ such that $w \geq_{\mathrm{A}} u$. However, there is some $i \in \mathbb{N}$ such that $(A_i, s_i) = (A, u)$. If A_i is predetermined by $w_0 w_1 \cdots w_{i-1}$, then u is ultimately periodic by Lemma 39, and hence $u \notin \mathcal{S}$. Otherwise property (6) contradicts $w = w_0 w_1 w_2 \cdots \geq_{\mathrm{A}} s_i = u$. Thus w has the required properties. $\square$

8. Future Work

Our results hint at an interesting structure of the transducer degrees of words $\langle p(n) \rangle$ for polynomials $p(n)$ of order $k \in \mathbb{N}$. Here, we have only scratched the surface of this structure. Many questions remain open, for example:

- (Q6) What is the structure of 'polynomial spiralling' degrees (depending on $k \in \mathbb{N}$)? Is the number of degrees finite for every $k \in \mathbb{N}$?

Q7 Are there interpolant degrees between the degrees of $\langle n^k \rangle$ and $\langle p_k(n) \rangle$?

Q8 Are there continuum many atoms?

Q9 Is the degree of the Thue–Morse sequence an atom?

References

[1] J.-P. Allouche and J. Shallit. *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge University Press, 2003.
[2] A. Belov. Some algebraic properties of machine poset of infinite words. *ITA*, 42(3):451–466, 2008.
[3] J. Berstel, L. Boasson, O. Carton, B. Petazzoni, and J.-É. Pin. Operations preserving regular languages. *Theoretical Computer Science*, 354(3):405–420, 2006.
[4] W. Bosma and H. Zantema. Ordering sequences by permutation transducers. *Indagationes Mathematicae*, 28(1):38–54, 2017.
[5] J. Endrullis, C. Grabmayer, D. Hendriks, and H. Zantema. The degree of squares is an atom. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of *LNCS*, pages 109–121. Springer, 2015.
[6] J. Endrullis and D. Hendriks. On Periodically Iterated Morphisms. In *Proc. Joint Meeting of Conference on Computer Science Logic and Symposium on Logic in Computer Science (CSL-LICS 2014)*, pages 39:1–39:10. ACM, 2014.
[7] J. Endrullis, D. Hendriks, and J. W. Klop. Degrees of streams. *Journal of Integers*, 11B(A6):1–40, 2011. Proceedings of the Leiden Numeration Conference 2010.
[8] J. Endrullis, J. Karhumäki, J.W. Klop, and A. Saarela. Degrees of infinite words, polynomials and atoms. In *Proc. Conf. Developments in Language Theory (DLT 2016)*, LNCS, pages 164–176. Springer, 2016.
[9] J. Endrullis, J. Karhumäki, J.W. Klop, and A. Saarela. Degrees of infinite words, polynomials and atoms (extended version). *CoRR*, abs/1803.03158, 2018.
[10] J. Endrullis, J.W. Klop, A. Saarela, and M. Whiteland. Degrees of transducibility. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of *LNCS*, pages 1–13. Springer, 2015.
[11] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, 1988. Reprint of the 1952 edition.
[12] K. Jacobs. *Invitation to mathematics*. Princeton University Press, 1992.
[13] B. Löwe. Complexity hierarchies derived from reduction functions. In *Classical and New Paradigms of Computation and their Complexity Hierarchies*, volume 23 of *Trends in Logic*, pages 1–14. Springer, 2004.
[14] P. Odifreddi. *Classical Recursion Theory*. Studies in logic and the foundations of mathematics. North-Holland, Amsterdam, 1999.
[15] G. Rayna. Degrees of finite-state transformability. *Information and Control*, 24(2):144–154, 1974.
[16] J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2003.
[17] A. Salomaa. *Jewels of Formal Language Theory*. Computer Science Press, Rockville, Maryland, 1981.
[18] J. R. Shoenfield. *Degrees of Unsolvability*. North-Holland, Elsevier, 1971.
[19] R. Van Wesep. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77*, pages 151–170. Springer, 1978.