PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

MAJORITY DIGRAPHS

TRI LAI, JÖRG ENDRULLIS, AND LAWRENCE S. MOSS

(Communicated by)

ABSTRACT. A *majority digraph* is a finite simple digraph $G = (V, \to)$ such that there exist finite sets A_v for the vertices $v \in V$ with the following property: $u \to v$ if and only if "more than half of the A_u are A_v". That is, $u \to v$ if and only if $|A_u \cap A_v| > \frac{1}{2} \cdot |A_u|$. We characterize the majority digraphs as the digraphs with the property that every directed cycle has a reversal. If we change $\frac{1}{2}$ to any real number $\alpha \in (0, 1)$, we obtain the same class of digraphs. We apply the characterization result to obtain a result on the logic of assertions "most X are Y" and the standard connectives of propositional logic.

1. INTRODUCTION

This paper poses a problem in combinatorics coming from logic. For finite sets X and Y, we say that *most X are Y* if $|X \cap Y| > \frac{1}{2}|X|$. If *most X are Y*, then it need not be the case that *most Y are X*, but it would follow (trivially) that *most X are X*. If *most X are Y* and *most Y are Z*, then it need not be the case that *most X are Z*. People with a background in logic would ask questions about sound inferences involving *most*: are there any interesting sound inferences at all? Is there a characterization of the collection of all sound inferences? What is the complexity of that collection? We shall formulate the inference question precisely and answer it in Section 4 near the end of this paper. The solution hinges on a result in elementary combinatorics, and this result is the main mathematical contribution of this paper.

If V is any finite set, and A_v is a finite set for $v \in V$, then we obtain a digraph $G = (V, \to)$ in a natural way: $u \to v$ iff *most A_u are A_v*. We are only interested in digraphs without self-loops, so when we write $u \to v$ in this paper, we tacitly assume that u and v are different. A *majority digraph* is a finite digraph isomorphic to some digraph of this form. The characterization of sound inferences involving *most* boils down to the characterization of majority digraphs. We next state our main result.

A *two-way edge* in a digraph is just an edge $u \to v$ in the digraph such that also $v \to u$. A *one-way edge* is an edge $u \to v$ such that $v \not\to u$. If G is a majority digraph via the sets A_v, and if there is a one-way edge from u to v, then $|A_v| > |A_u|$. Thus G cannot have *one-way cycles*: there are no paths

$$v_1 \to v_2 \to \cdots \to v_n = v_1 \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 05C62, 03B65.

This work was partially supported by a grant from the Simons Foundation (#245591 to Lawrence Moss).

such that for $1 \leq i < n$, $v_{i+1} \not\to v_i$. (There may be cycles with two-way edges.) This point was noticed by Chloe Urbanski [U], and she conjectured that the absence of one-way cycles characterizes majority digraphs. This turns out to be true, and it is our main result.

More generally, for any $\alpha \in (0,1)$, we say that $G = (V, \to)$ is a *proportionality α-digraph* if there exist finite sets A_v for $v \in V$ with the following property:

$$u \to v \quad \text{iff} \quad |A_u \cap A_v| > \alpha \cdot |A_u|.$$

So a majority digraph is a proportionality $\frac{1}{2}$-digraph. Our second main result is that the characterization result for majority digraphs holds as well for proportionality α-digraphs, for all $\alpha \in (0,1)$.

1.1. Contents.

Section 1.2 has a very general (and very easy) representation result on digraphs with the property that every directed cycle has a *reversal*. That is, for every path as in (1.1) there is some $1 \leq i < n$ such that $v_{i+1} \to v_i$. (This is the same as having no one-way cycles.) In Section 2, we show that this condition characterizes majority digraphs; indeed, it characterizes α-proportionality digraphs for all rational α. Then in Section 3 we obtain the result for all real $\alpha \in (0,1)$. The work on rational p/q in Section 2 is not merely a special case of the later results on real α in Section 3. The point is that to carry out our construction for irrational α necessitates using much larger sets than the construction when α is a rational number. Put differently, the result in Theorem 3.3 is a generalization of the result in Theorem 2.2, but the construction in Theorem 2.2 gives better bounds for the digraphs it constructs.

We conclude the paper in Section 4 by returning to the matter in logic with which we began. Section 4 may be read after Section 2.

1.2. Preliminary.

For a fixed number n, an *appropriate pair* is a pair (S,T) such that

1. S is a set of unordered pairs $\{i,j\}$ from the set of numbers $\{1,\ldots,n\}$.
2. T is a set of ordered pairs (i,j) from $\{1,\ldots,n\}$.
3. If $(i,j) \in T$, then $i < j$.
4. If $(i,j) \in T$, then $\{i,j\} \notin S$.

Further, every appropriate pair determines a digraph $G_{S,T}$. The vertices of $G_{S,T}$ are the points of $\{1,\ldots,n\}$, and we put $i \to j$ iff either $\{i,j\} \in S$ or $(i,j) \in T$.

Proposition 1.1. *Let G be a digraph on n vertices with no one-way cycles. Then there is an appropriate pair (S,T) such that G is isomorphic to $G_{S,T}$.*

Proof. First, we may assume that the vertices of G are $\{1,\ldots,n\}$. We may list these in topological order. So we have a sequence $1,\ldots,n$, with the property that if $i \to j$ but $j \not\to i$, then $i < j$. This is due to the assumption that there be no one-way cycles. We can take S to be the set of pairs corresponding to the two-way edges, and T the one-way edges. □

2. The case when α is a rational number p/q

In this section, we represent digraphs with no one-way cycles as proportionality p/q-digraphs for all natural numbers $0 < p < q$. Taking $p/q = 1/2$, we see that digraphs with no one-way cycles are majority digraphs.

The reader may wish to consult a worked example which we present in Section 2.1 below.

For a sequence of sets $A_1, \ldots, A_n$ and for $1 \leq i < j \leq n$, we write $A_i \sqcap A_j$ for $(A_i \cap A_j) \setminus \bigcup_{k \neq i,j} A_k$. We call this the *private intersection* of A_i and A_j. This is an example of what is sometimes called a *zone* in a Venn diagram.

Lemma 2.1. *Let $p \leq q$ be natural numbers. For all n, there are sets $B_1, \ldots, B_n$ such that*

(1) $|B_i| = pq^{n-1}$.
(2) *For $i \neq j$, $|B_i \cap B_j| = p^2 q^{n-2} = \frac{p}{q}|B_i|$.*
(3) *For $i \neq j$, $|B_i \sqcap B_j| = p^2(q-p)^{n-2}$.*

Proof. Consider $S = \{1, \ldots, q\}$. Let

$$B_i \quad = \quad \{(s_1 s_2 \cdots s_i \cdots s_n) \in \{1, \ldots, q\}^n : s_i \in \{1, \ldots, p\}\}.$$

The first two parts are obvious. $B_i \sqcap B_j$ is the set of sequences $(s_1 s_2 \cdots s_i \cdots s_n)$ so that s_i and s_j belong to $\{1, \ldots, p\}$, and the other entries do not belong to this set. □

We shall use the following elementary observation:

$$\frac{p + ar}{q + ar + s} > \frac{p}{q} \quad \text{iff} \quad a > \frac{ps}{r(q-p)} \ . \tag{2.1}$$

Theorem 2.2. *Let G be a digraph on n vertices with no one-way cycles. Let $p < q$ be positive natural numbers. Then G is a proportionality p/q-digraph.*

Proof. By Proposition 1.1, we find an appropriate pair (S, T) such that G is isomorphic to $G_{S,T}$.

For our p and q, let $B_1, \ldots, B_n$ be as in Lemma 2.1. We shall modify these sets in several steps to obtain sets $A_1, \ldots, A_n$ such that the following hold:

(a) If $\{i, j\} \in S$, then $|A_i \cap A_j| > \frac{p}{q}|A_i|$ and $|A_i \cap A_j| > \frac{p}{q}|A_j|$.
(b) If $i < j$ and $(i, j) \in T$, then $|A_i \cap A_j| > \frac{p}{q}|A_i|$ but $|A_i \cap A_j| \leq \frac{p}{q}|A_j|$.
(c) If $i < j$ and $(i, j) \notin T$, then $|A_i \cap A_j| \leq \frac{p}{q}|A_i|$ and $|A_i \cap A_j| \leq \frac{p}{q}|A_j|$.

Let a and m be natural numbers which are large enough so that the following hold:

$$(q - p)a \quad > \quad q\,; \tag{2.2}$$

$$mp^2(q-p)^{n-2} \quad > \quad apn. \tag{2.3}$$

To begin, take m copies of all points in all sets B_i. (That is, let $A_i = B_i \times \{1, \ldots, m\}$.) This arranges that $|A_i| = mpq^{n-1}$ for all i, and for $i \neq j$, $|A_i \cap A_j| = mp^2 q^{n-2}$, and $|A_i \sqcap A_j| > apn$. We have used (2.3) here.

Add a single set C of apn points simultaneously to all A_i. That is, we have $C \subseteq \bigcap_i A_i$. (We are going to continue to call the sets A_i rather than change the notation.) This adds apn points to all intersections $A_i \cap A_j$, so now these sets have size $mp^2 q^{n-2} + apn$. But this addition leaves all private intersections $A_i \sqcap A_j$ unchanged.

Next, for each i, add qi fresh points to A_i. In this step, we add different points to the different A_i. This step does not change (private) intersections, it only increases the sizes of the sets.

When $\{i,j\} \in S$, the rest of our construction will not alter the intersection $A_i \cap A_j$ or the sizes of A_i and A_j. So in this case, we shall have at the end that

$$\frac{|A_i \cap A_j|}{|A_i|} \quad = \quad \frac{mp^2q^{n-2} + apn}{mpq^{n-1} + apn + qi} \quad \geq \quad \frac{mp^2q^{n-2} + apn}{mpq^{n-1} + apn + qn} \quad > \quad \frac{p}{q} \; .$$

We have used (2.1) with $r = pn$ and $s = qn$, and also the assumption (2.2) on a. Similarly, $|A_i \cap A_j| > \frac{p}{q}|A_j|$.

We are left with two cases: (a) $i < j$ and $(i,j) \in T$ (and thus $\{i,j\} \notin S$) and (b) $i < j$ and $(i,j) \notin T$ and $\{i,j\} \notin S$. For the pairs of the first type, we make a certain adjustment to the sets we have, removing points from $A_i \sqcap A_j$ and returning them as separate copies in the two sets. (So this type of adjustment does not change the size of any A_i, but it decreases the sizes of the intersections $A_i \cap A_j$.) It will turn out that the number of points which we remove in this case depends on i. And for the second type, we remove *all* the points in $A_i \sqcap A_j$ and return them as separate copies in the two sets. All of these adjustments of either type may be carried out at the same time, and there is no need to order them.

The case (b) of $i < j$ and also $(i,j) \notin T$ and $\{i,j\} \notin S$ is easier to handle, so let us look at this first. The private intersection $A_i \sqcap A_j$ has size $mp^2(q-p)^{n-2}$. Let us call this number z. Take the entire private intersection and remove it, returning separate copies of the same size z to A_i and to A_j. The removal decreased the size of the intersection $A_i \cap A_j$ by z. By (2.3), $apn - z < 0$. We calculate:

$$\frac{|A_i \cap A_j|}{|A_i|} \quad = \quad \frac{mp^2q^{n-2} + apn - z}{mpq^{n-1} + apn + qi} \quad < \quad \frac{mp^2q^{n-2}}{mpq^{n-1}} \quad = \quad \frac{p}{q} \; .$$

Similarly, $|A_i \cap A_j|/|A_j| < p/q$.

Finally, let $i < j \leq n$ and $(i,j) \in T$. The idea is to do something similar to what we did in the last paragraph: remove a certain number of points from the private intersection $A_i \sqcap A_j$ and then return the same number of points in separate copies to A_i and A_j. But we want to remove a *proper* subset of points, so that more than p/q of the A_i are A_j, but at most p/q of the A_j are A_i. By (2.2),

$$\frac{p}{q}apn + pi \quad < \quad \frac{p}{q}apn + pn \quad < \quad apn \; .$$

Let

$$c \quad = \quad \left\lceil \left(\frac{q-p}{q}\right) apn \right\rceil - pi - 1 \; .$$

This has the property that

$$\frac{p}{q}apn + pi \quad < \quad apn - c \quad \leq \quad \frac{p}{q}apn + pi + 1 \; .$$

Note that $c < apn$, and as we have seen, $apn < |A_i \sqcap A_j|$. We remove c points from $A_i \sqcap A_j$ and return them separately to A_i and A_j. So the intersection $A_i \cap A_j$ has size $mp^2q^{n-2} + apn - c$. To check that this works, we calculate:

$$\begin{aligned}
\frac{p}{q}(mpq^{n-1} + apn + qi) &= mp^2q^{n-2} + \frac{p}{q}apn + pi \\
&< mp^2q^{n-2} + apn - c \\
&\leq mp^2q^{n-2} + \frac{p}{q}apn + pi + 1 \\
&\leq mp^2q^{n-2} + \frac{p}{q}apn + pj \qquad \text{(since } i < j \text{ and } 1 \leq p) \\
&= \frac{p}{q}(mpq^{n-1} + apn + qj).
\end{aligned}$$

That is,

$$\frac{p}{q}|A_i| \quad < \quad |A_i \cap A_j| \quad \leq \quad \frac{p}{q}|A_j|.$$

We have achieved our goals (a), (b), and (c). This completes the proof. □

Remark Let us see how many points are needed to exhibit a digraph without one-way cycles as a $\frac{1}{2}$-digraph. Suppose that G has n vertices and e edges. We would like to know the size of $\bigcup_i A_i$ using the method of this section. We have $p = 1$ and $q = 2$, and in Lemma 2.1, $\bigcup_i A_i$ has size 2^n. Further, we may take $a = 3$ and $m = 3n$. Following the proof, we get a universe of at most $3n(1+2^n)+n(n+1)+3ne$ points. So we get $|\bigcup_i A_i| = O(n2^n)$.

2.1. **Example.** We illustrate all of the ideas in the proof of Theorem 2.2 with an example. Consider the digraph G shown in Figure 1 below. We thus begin with $n = 4$, $p = 1$, and $q = 2$. The usual order $1 < 2 < 3 < 4$ has the property that if $i \to j$ but $j \not\to i$, then $i < j$. From the graph, we have

$$\begin{aligned}
S &= \{\{1, 2\}, \{2, 3\}\} \\
T &= \{(1, 3), (3, 4), (2, 4)\}
\end{aligned}$$

Lemma 2.1 gives sets B_1, …, B_4 with the property that all zones in their Venn diagram have size 1. This is the first diagram in Figure 2. Continuing, we take a to be 3; this is the minimum number so that (2.2) holds. Thus $apn = 12$, and $m = 13$ is the smallest so that (2.3) holds. We continue by taking 13 copies of all points, and we rename the sets A_1, …, A_4.

Next, we add $apn = 12$ points to $\bigcap A_i$. This is shown in the left-hand Venn diagram on the second row. Continuing, we add 2 points to A_1, 4 to A_2, 6 to A_3, and 8 to A_4. When we add to A_i in this step, we are adding to $A_i \setminus \bigcup_{j \neq i} A_j$. This is what we mean by a "private" addition. This is shown in the right-hand Venn diagram on the second row.

We can check at this point that for $\{i, j\} \in S$, $|A_i \cap A_j| > \frac{1}{2}|A_j|$. For example, when $j = 3$, and $i = 2$, we have $|A_3 \cap A_2| = 64$, and $A_2 = 120$.

Next, $(1, 4) \notin T$ and $\{1, 4\} \notin S$. So we take $A_1 \sqcap A_4$, remove all 13 of its points, and then add 13 points privately to A_1, and finally 13 other points privately to A_4. The picture is the left diagram on the bottom row. Then $|A_1 \cap A_4| = 51$. This is smaller than $\frac{1}{2}|A_1| = 59$ and also smaller than $\frac{1}{2}|A_4| = 62$.

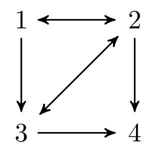

FIGURE 1. A digraph without one-way cycles used to illustrate the steps in Theorem 2.2.

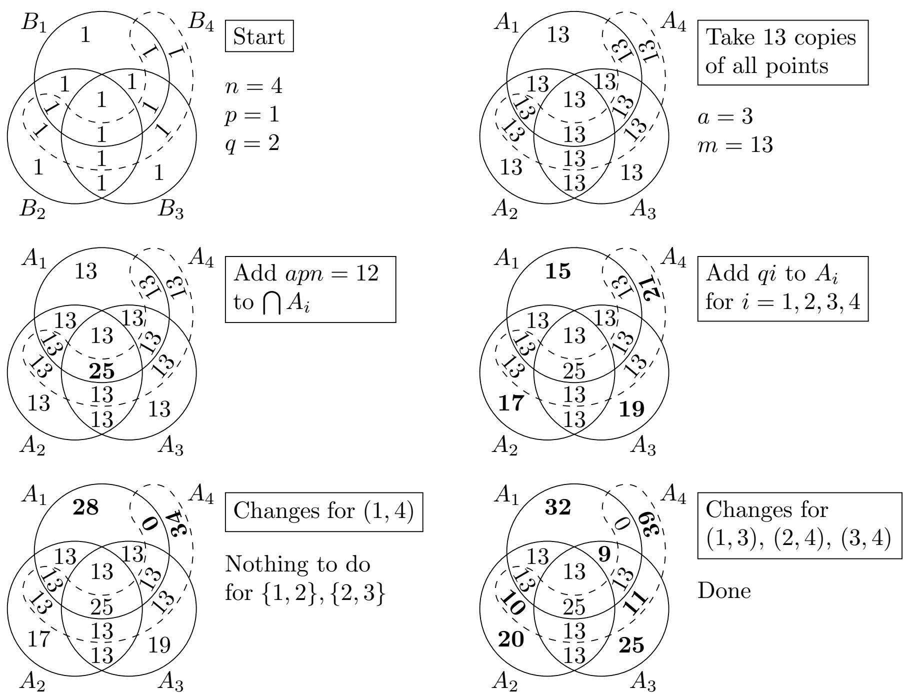

FIGURE 2. Six Venn diagrams illustrating the construction. These are explained in detail in Section 2.1.

It remains to take care of the pairs in T: $(1, 3)$, $(3, 4)$, and $(2, 4)$. For $(1, 3)$, the value of c is $\lceil \frac{1}{2}(12) \rceil - 1(1) - 1 = 4$. We remove 4 points from $A_1 \sqcap A_3$ and return them privately to A_1 and A_3. This is how we get $|A_1 \sqcap A_3| = 13 - 4 = 9$ at the end. For $(3, 4)$, $c = \lceil \frac{1}{2}(12) \rceil - 1(3) - 1 = 2$. We remove 2 points from $A_3 \sqcap A_4$ and return them privately to A_3 and A_4. And at the end, $|A_3 \sqcap A_4| = 13 - 2 = 11$. For $(2, 4)$, $c = 3$. We remove 3 points from $A_2 \sqcap A_4$ and return them privately to A_2 and A_4. Thus $|A_2 \sqcap A_4| = 13 - 3 = 10$. Then totaling up the three private additions mentioned in this paragraph gives the sizes of the sets $A_i \setminus \bigcup_{j \neq i} A_j$ for $i = 1, \ldots 4$. The final Venn diagram is shown at the bottom right of Figure 2. It exhibits the digraph in Figure 1 as a majority digraph.

3. Proportionality α-digraphs

This section proves Theorem 3.3, a generalization of Theorem 2.2. We begin with a few auxiliary definitions. Fix a natural number n, and let P be the set of all subsets of $\{1, \ldots, n\}$. We define

$$\Delta(i, \ldots, i_\ell) = \Delta(\{i, \ldots, i_\ell\}) = \{\, I \in P \mid \ \{i, \ldots, i_\ell\} \subseteq I\}.$$

As the notation indicates, in this discussion we frequently omit set braces in the arguments of Δ. An *n-size function* f is a function $f : P \to \mathbb{R}_+$, and we associate to f the function $f^* : P \to \mathbb{R}_+$ defined by:

$$f^*(I) = \sum_{J \in \Delta(I)} f(J) \tag{3.1}$$

for all $I \in P$. The intuition is that $f(I)$ indicates the size of the private intersection

$$\bigcap_{i \in I} A_i \setminus \bigcup_{j \notin I} A_j \ ,$$

while $f^*(I)$ indicates the size of the intersection $\bigcap_{i \in I} A_i$.

Here are the canonical examples of size functions.

Lemma 3.1. *Let $n \in \mathbb{N}$ and $\alpha \in (0,1)$. There exists a size function $f : P \to \mathbb{R}_+$ such that*

1. *$f^*(i) = 1$ for every $1 \le i \le n$.*
2. *$f(i,j) = \alpha(1-\alpha)^{n-2}$ for every $1 \le i, j \le n$ with $i \ne j$.*
3. *$f^*(i,j) = \alpha$ for every $1 \le i, j \le n$ with $i \ne j$.*

Proof. We define $f : P \to \mathbb{R}_+$ by:

$$f(I) = \alpha^{|I|-1}(1-\alpha)^{n-|I|}$$

for all $I \subseteq P$. Then for $1 \le i \le n$ we have

$$\begin{aligned}
f^*(i) = \sum_{J \in \Delta(i)} f(J) &= \sum_{J \in \Delta(i)} \left(\alpha^{|J|-1}(1-\alpha)^{n-|J|}\right) \\
&= \sum_{\ell=1}^{n} \left(\binom{n-1}{\ell-1} \alpha^{\ell-1}(1-\alpha)^{n-\ell}\right) \\
&= \sum_{\ell=0}^{n-1} \left(\binom{n-1}{\ell} \alpha^{\ell}(1-\alpha)^{(n-1)-\ell}\right) \\
&= (\alpha + (1-\alpha))^{n-1} = 1.
\end{aligned}$$

For $1 \le i, j \le n$ with $i \ne j$ we get $f(i,j) = \alpha(1-\alpha)^{n-2}$ by definition, and:

$$\begin{aligned}
f^*(i,j) = \sum_{J \in \Delta(i,j)} f(J) &= \sum_{J \in \Delta(i,j)} \left(\alpha^{|J|-1}(1-\alpha)^{n-|J|}\right) \\
&= \sum_{\ell=2}^{n} \left(\binom{n-2}{\ell-2} \alpha^{\ell-1}(1-\alpha)^{n-\ell}\right) \\
&= \alpha \cdot \sum_{\ell=0}^{n-2} \left(\binom{n-2}{\ell} \alpha^{\ell}(1-\alpha)^{(n-2)-\ell}\right) \\
&= \alpha \cdot (\alpha + (1-\alpha))^{n-2} = \alpha.
\end{aligned}$$

This concludes the proof. □

Lemma 3.2. *Let G be a digraph on n vertices. Let $f : P \to \mathbb{R}_+$ be an n-size function such that*

$$\begin{aligned} i \to j &\implies f^*(i,j) > \alpha \cdot f^*(i) \\ i \not\to j &\implies f^*(i,j) < \alpha \cdot f^*(i) \end{aligned}$$

for all vertices $i \neq j$ of G. Then G is a proportionality α-digraph.

Proof. Let $\varepsilon > 0$ be a real number small enough such that for all $1 \leq i, j \leq n$, $i \neq j$:

$$\frac{f^*(i,j)}{f^*(i)} < \alpha \implies \frac{f^*(i,j)}{f^*(i) - \varepsilon} < \alpha; \tag{3.2}$$

$$\frac{f^*(i,j)}{f^*(i)} > \alpha \implies \frac{f^*(i,j) - \varepsilon}{f^*(i)} > \alpha. \tag{3.3}$$

We choose $N \in \mathbb{N}$ such that

$$\frac{2^{n-1}}{N} < \varepsilon. \tag{3.4}$$

For every $J \in P$, let $A(J)$ be a set of $\lfloor f(J) \cdot N \rfloor$ points, with $A(J) \cap A(J') = \emptyset$ for $J \neq J'$. Then

$$f(J) \cdot N - 1 \quad < \quad |A(J)| \quad \leq \quad f(J) \cdot N. \tag{3.5}$$

For $1 \leq i \leq n$ we define the set A_i as follows:

$$A_i = \bigcup_{J \in \Delta(i)} A(J).$$

Then by (3.1) and (3.5) it follows for $1 \leq i \leq n$ that:

(3.6)

$$f^*(i) \cdot N - 2^{n-1} = \sum_{J \in \Delta(i)} (f(J) \cdot N - 1) \quad < \quad \underbrace{\sum_{J \in \Delta(i)} |A(J)|}_{=|A_i|} \quad \leq \quad f^*(i) \cdot N.$$

Similarly, for $1 \leq i < j \leq n$ we have:

(3.7)

$$f^*(i,j) \cdot N - 2^{n-2} = \sum_{J \in \Delta(i,j)} (f(J) \cdot N - 1) \quad < \quad \underbrace{\sum_{J \in \Delta(i,j)} |A(J)|}_{=|A_i \cap A_j|} \quad \leq \quad f^*(i,j) \cdot N.$$

Now from (3.6) and (3.7) we conclude:

$$\frac{f^*(i,j) \cdot N - 2^{n-2}}{f^*(i) \cdot N} \quad < \quad \frac{|A_i \cap A_j|}{|A_i|} \quad < \quad \frac{f^*(i,j) \cdot N}{f^*(i) \cdot N - 2^{n-1}},$$

and hence

(3.8)

$$\frac{f^*(i,j) - \varepsilon}{f^*(i)} \leq \frac{f^*(i,j) - \frac{2^{n-2}}{N}}{f^*(i)} \quad < \quad \frac{|A_i \cap A_j|}{|A_i|} \quad < \quad \frac{f^*(i,j)}{f^*(i) - \frac{2^{n-1}}{N}} \leq \frac{f^*(i,j)}{f^*(i) - \varepsilon}.$$

Now for all vertices $i \neq j$ of G we have:

$$i \to j \implies f^*(i,j) > \alpha \cdot f^*(i) \overset{\text{by (3.3) and (3.8)}}{\implies} \alpha < \frac{f^*(i,j) - \varepsilon}{f^*(i)} < \frac{|A_i \cap A_j|}{|A_i|};$$

$$i \not\to j \implies f^*(i,j) < \alpha \cdot f^*(i) \overset{\text{by (3.2) and (3.8)}}{\implies} \alpha > \frac{f^*(i,j)}{f^*(i) - \varepsilon} > \frac{|A_i \cap A_j|}{|A_i|}.$$

Hence G is a proportionality α-digraph. ☐

The remainder of the section is concerned constructs of the appropriate size function for a digraph G with no one-way cycles.

Theorem 3.3. *If G has no one-way cycles, then G is a proportionality α-digraph.*

Proof. By Proposition 1.1 there is an appropriate pair (S,T) such that G is isomorphic to $G_{S,T}$. Without loss of generality, assume that $G = G_{S,T}$.

We need the following variant of (2.1):

$$\frac{\alpha + \varepsilon}{1 + \varepsilon + \delta} > \alpha \quad \text{iff} \quad \frac{\varepsilon(1-\alpha)}{\alpha} > \delta\ . \tag{3.9}$$

Let $f : P \to \mathbb{R}_+$ be as in Lemma 3.1, where we take n to be the number of vertices in G, and α as in our theorem. Let ε and δ be defined as follows:

$$\varepsilon = \frac{\alpha(1-\alpha)^{n-1}}{2} \qquad\qquad \delta = \frac{\varepsilon(1-\alpha)}{2\alpha}.$$

Roughly speaking, we add ε commonly to the intersection of all vertices, and $\frac{i\delta}{n}$ privately to vertex i for all i. Formally, we define a size function $g : P \to \mathbb{R}_+$ by:

$$\begin{aligned} g(1,\ldots,n) &= f(1,\ldots,n) + \varepsilon \\ g(i) &= f(i) + \frac{i\delta}{n} && \text{for all } 1 \le i \le n; \\ g(I) &= f(I) && \text{for all } I \in P \text{ with } 1 < |I| < n. \end{aligned}$$

Then we have for all $1 \le i, j \le n$ with $i \neq j$:

$$\begin{aligned} g^*(i) &= 1 + \varepsilon + \frac{i\delta}{n}; \\ g^*(i,j) &= \alpha + \varepsilon. \end{aligned}$$

By (3.9), we have that for all $1 \le i, j \le n$ with $i \neq j$

$$\frac{g^*(i,j)}{g^*(i)} = \frac{\alpha + \varepsilon}{1 + \varepsilon + \frac{i\delta}{n}} \ge \frac{\alpha + \varepsilon}{1 + \varepsilon + \delta} > \alpha. \tag{3.10}$$

As a consequence, for each $1 \le i < j \le n$ there exists $\gamma(i,j) \in [0, \varepsilon]$ such that

$$\begin{aligned} &\frac{g^*(i,j) - \gamma(i,j)}{g^*(i)} = \frac{\alpha + \varepsilon - \gamma(i,j)}{1 + \varepsilon + \frac{i\delta}{n}} > \alpha \\ \text{and} \quad &\frac{g^*(i,j) - \gamma(i,j)}{g^*(j)} = \frac{\alpha + \varepsilon - \gamma(i,j)}{1 + \varepsilon + \frac{j\delta}{n}} < \alpha. \end{aligned} \tag{3.11}$$

Now define for all $1 \leq i < j \leq n$:

$$h(i) = g(i) + \sum_{j=1}^{n} \varphi(i,j) \qquad \varphi(i,j) = \begin{cases} 0 & \text{if } i \to j \text{ and } j \to i; \\ g(i,j) & \text{if } i \not\to j \text{ and } j \not\to i; \\ \gamma(i,j) & \text{if } i \to j \text{ and } j \not\to i. \end{cases}$$

$$h(i,j) = g(i,j) - \varphi(i,j)$$

$$h(I) = g(I), \text{ for all other } I$$

We only define $h(i,j)$ when $i < j$. Note that $h^*(i) = g^*(i)$ for every $1 \leq i \leq n$, as we we add privately to A_i as much as we remove from the private intersections $A_i \sqcap A_j$.

We check the hypotheses of Lemma 3.2 for h. First, if $i \to j$ and $j \to i$, then

$$\frac{h^*(i,j)}{h^*(i)} = \frac{g^*(i,j)}{g^*(i)} > \alpha.$$

Second, if $i \not\to j$ and $j \not\to i$, then $h^*(i,j) + g(i,j) = g^*(i,j)$. Thus

$$\frac{h^*(i,j)}{h^*(i)} = \frac{g^*(i,j) - g(i,j)}{g^*(i)} = \frac{\alpha + \varepsilon - \alpha(1-\alpha)^{n-2}}{1 + \varepsilon + \frac{i\delta}{n}} < \frac{\alpha}{1 + \varepsilon + \frac{i\delta}{n}} < \alpha.$$

Finally, if $i \to j$ and $j \not\to i$, then $i < j$. And by (3.11),

$$\frac{h^*(i,j)}{h^*(i)} = \frac{g^*(i,j) - \gamma(i,j)}{g^*(i)} > \alpha,$$

$$\frac{h^*(i,j)}{h^*(j)} = \frac{g^*(i,j) - \gamma(i,j)}{g^*(j)} < \alpha.$$

This completes the proof. □

Remark Let G be a digraph on n points with no one-way cycles. If $\alpha \approx \frac{1}{2}$, then the method of Theorem 3.3 represents a digraph G a proportionality $\frac{1}{2}$-digraph with $|\bigcup A_g| = O(n^2 2^{2n})$. Here is the reasoning.

(1) Let $\alpha \sim 1/2$. Then by Lemma 3.1, we have $f(I) = (1/2)^{n-1}$, for any I.
(2) In the proof of Theorem 3.3. with $\alpha \sim 1/2$, $\varepsilon = (1/2)^{n+1}$ and $\delta = (1/2)^{n+2}$.
(3) The maximum value of $g(I)$ is at most

$$\max\ f(I) + \varepsilon\ (1/2)^{n-1} + (1/2)^{n+1} = 5(1/2)^{n+1}.$$

(4) Next, we estimate size of $h(I)$. It is less than

$$\max g(I) + n \times (\max \varphi(i,j)) \sim 5(1/2)^{n+1} + n(1/2)^{n-1} = (4n+5)(1/2)^{n+1}.$$

(5) Now $h(I)$ acts like the function $f(I)$ in Lemma 3.2. We take $N = 2^{2n}$. We have $|A(J)| < \max h(I) \times N \sim (4n+5)2^{n-1}$.
(6) We have

$$|A(i)| < (2^n) \times (4n+5)2^{n-1} < (4n+5)2^{2n}.$$

Therefore, the size of $\bigcup A_i$ is less than $(4n^2 + 5n)2^{2n}$.

At the end of Section 2, we saw that the method of Theorem 2.2 represents G as a majority digraph with $|\bigcup A_g| = O(n2^n)$.

However, even though this suggests that Theorem 3.3 is not as good a result as Theorem 2.2 we emphasize that Theorem 3.3 works for all real α. We do not know how to extend the construction in Theorem 2.2 to work on all real α.

4. Application: the boolean logic of "most X are Y"

We have characterized the α-proportionality digraphs as those with no one-way cycles. In particular, when $\alpha = 1/2$, we see that every digraph with no one-way cycles is a majority digraph. We conclude with an application of this last result in logic. What we discuss would be called a completeness theorem for the boolean logic of "most X are Y". We start with a collection of *one-place relation symbols* X, Y, Z, ... We then form *atomic sentences* of the form $\mathsf{M}(X, Y)$. (Note that X and Y may be the same symbol here. Up until now in this paper, we mainly worried about such sentences when X and Y are *different*. So we have a slight complication to keep in mind.) $\mathsf{M}(X, Y)$ is an abbreviation for Most X are Y. Finally, we form *sentences* from atomic sentences using the *boolean connectives* of propositional logic, namely negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\to$) and bi-implication ($\leftrightarrow$). So as just one example of a sentence, we would have

$$(\mathsf{M}(X, Y) \wedge \neg\mathsf{M}(X, Z)) \vee \mathsf{M}(Y, X)\ .$$

We call this logical language $\mathcal{L}(\mathsf{most})$. We are interested in the problem of *inference* in $\mathcal{L}(\mathsf{most})$. To formulate this precisely, we need the notion of *semantics*. For this, we use *models*. A model of $\mathcal{L}(\mathsf{most})$ is a structure $\mathcal{U} = (U, [\![\ \ ]\!])$ consisting of a finite set U together with *interpretations* $[\![X]\!] \subseteq U$ of each one-place relation symbol X. We then interpret our sentences in $\mathcal{U}$ as follows

$$\mathcal{U} \models \mathsf{M}(X, Y) \quad \text{iff} \quad |[\![X]\!] \cap [\![Y]\!]| > \tfrac{1}{2}|[\![X]\!]|\ .$$

We also read "$\mathcal{U} \models \mathsf{M}(X, Y)$" as "in the model $\mathcal{U}$, most X's are Y's." If it is not the case that $\mathcal{U} \models \mathsf{M}(X, Y)$, then we write $\mathcal{U} \not\models \mathsf{M}(X, Y)$. Observe that if $[\![X]\!]$ or $[\![Y]\!]$ is empty in a given model, then automatically $\mathcal{U} \not\models \mathsf{M}(X, Y)$.

We use φ and ψ as variables ranging over sentences in $\mathcal{L}(\mathsf{most})$, and Γ as a variable denoting arbitrary finite sets of sentences. Sentences with boolean connectives are given truth values in the usual way. For example,

$$\begin{array}{lll} \mathcal{U} \models \neg\varphi & \text{iff} & \mathcal{U} \not\models \varphi \\ \mathcal{U} \models \varphi \wedge \psi & \text{iff} & \mathcal{U} \models \varphi \text{ and } \mathcal{U} \models \psi \end{array}$$

We say that $\mathcal{U} \models \Gamma$ if $\mathcal{U} \models \psi$ for all $\psi \in \Gamma$. The main semantic definition is:

$$\Gamma \models \varphi \text{ if for all finite models } \mathcal{U}, \text{ if } \mathcal{U} \models \Gamma, \text{ then } \mathcal{U} \models \varphi.$$

This relation $\Gamma \models \varphi$ between finite sets of sentences and single sentences is called the *consequence relation* of the logic. Up until now, we have a semantic definition, having to do with all possible models of $\mathcal{L}(\mathsf{most})$. We shall define a proof-theoretic notion $\Gamma \vdash \varphi$ and then proving the Soundness/Completeness Theorem: $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.

One important feature of our semantics is that we restrict attention to *finite* sets, contrary to the usual practice in logic. This is because we want to work with numerical proportions.

The logical system that we use is defined in Figure 3. By *propositional tautologies* we mean substitution instances of propositional tautologies. The next axiom just says that if $\mathsf{M}(X, Y)$ in a given model, then X and Y must be non-empty in the model. (Incidentally, in this discussion one should be sure to note the difference between two uses of the $\to$ symbol: one for the edges in a digraph, and the other for a connective in $\mathcal{L}(\mathsf{most})$.) Consequently, in the same model we have $\mathsf{M}(X, X)$ and also $\mathsf{M}(Y, Y)$. This tells us that our axiom is *sound*.

Axioms	all propositional tautologies
	$\mathsf{M}(X,Y) \to (\mathsf{M}(X,X) \wedge \mathsf{M}(Y,Y))$
	$(\mathsf{M}(X_1,X_2) \wedge \mathsf{M}(X_2,X_3) \wedge \cdots \wedge \mathsf{M}(X_n,X_1))$ $\to (\mathsf{M}(X_2,X_1) \vee \mathsf{M}(X_3,X_2) \vee \cdots \vee \mathsf{M}(X_1,X_n))$
Inference Rule	from $\varphi \to \psi$ and φ, infer ψ (Modus Ponens)

FIGURE 3. The logical system for $\mathcal{L}(\mathsf{most})$, the boolean logic of "most X are Y".

The key feature of the system is the infinite collection of axioms which together say that every cycle in the "most" relation has a reversal. As we now know, this characterizes majority digraphs. Our logical result is in essence a logical reformulation of this digraph-theoretic representation theorem.

We say that $\Gamma \vdash \varphi$ if there is a finite sequence of sentences such that each is either a member of Γ or an axiom, or else comes from earlier sentences in the sequence using the one rule of the system, Modus Ponens.

As an example, the reader familiar with propositional logic will easily see that

$$\mathsf{M}(X,Y), \mathsf{M}(Y,Z) \vdash \neg\mathsf{M}(Z,X) \vee \mathsf{M}(Y,X) \vee \mathsf{M}(Z,Y) \vee \mathsf{M}(X,Z).$$

Theorem 4.1. *For all finite sets $\Gamma \cup \{\varphi\}$ of sentences in $\mathcal{L}(\mathsf{most})$, $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$.*

Remark The completeness half of this result is false if we allow Γ to be infinite. The reason is that if we take

$$\Gamma \quad = \quad \{\mathsf{M}(X_{i+1},X_i) : i = 1,2,\ldots\} \cup \{\neg\mathsf{M}(X_i,X_{i+1}) : i = 1,2,\ldots\},$$

then we have $\Gamma \models \varphi$ for all φ, even when φ is a contradictory sentence such as $\mathsf{M}(X,X) \wedge \neg\mathsf{M}(X,X)$. (To see this, suppose that $\mathcal{U}$ satisfies every sentence in Γ. Then $|[\![X_1]\!]| > |[\![X_2]\!]| > \cdots$. It follows that there are no finite models of Γ. And since our semantics is only concerned with finite models, it follows that $\Gamma \models \varphi$ for all φ.) But for a contradictory φ, $\Gamma \not\vdash \varphi$: proofs are finite, and it is easy to see from the soundness that no finite subset of Γ can derive a contradiction.

Proof. Using standard facts, we may restrict attention to the case when Γ is the empty set. In effect, we can move sentences across both relations $\models$ and $\vdash$. So we are left to prove that $\vdash \varphi$ if and only if $\models \varphi$. In words, φ has a proof in our system if and only if φ is true in all models.

The soundness part is a routine induction on the lengths of proofs in the system, and we are going to omit these details. In fact, soundness of a logical system is a very weak property, and the main point of interest is the completeness of the system. We argue for an equivalent assertion: if φ is *consistent* in the logic (that is, if $\not\vdash \neg\varphi$), then there is some (finite) model of φ.

Let $\mathcal{F}$ be the finite set of one-place relational symbols X, Y, ..., which occur in φ. Using the propositional part of the logic, we may assume that our consistent sentence φ is in *disjunctive normal form over $\mathcal{F}$*. That is, φ may be written as $\psi \vee \cdots \vee \psi_n$, where $n \geq 1$ and (1) each ψ_i is a conjunction of atomic sentences and their negations; (2) for all $X, Y \in \mathcal{F}$, ψ_i either contains $\mathsf{M}(X,Y)$ as a conjunct, or else it contains $\neg\mathsf{M}(X,Y)$ as a conjunct; (3) each ψ_i is consistent in the logic. We show that ψ has a model. (The same holds for the other ψ_i.) Then a model of ψ is a model of φ, and we are done.

Let

$$G \quad = \quad \{X : \mathsf{M}(X,X) \text{ is a conjunct of } \psi\},$$

And make G into a simple digraph by setting (for $X \neq Y$)

$$X \to Y \text{ in } G \quad \text{iff} \quad \mathsf{M}(X,Y) \text{ is a conjunct of } \psi.$$

We claim that every cycle in G has a reversal. For suppose that in G,

$$Z_1 \to Z_2 \to Z_3 \to \cdots \to Z_n \to Z_1 \ .$$

Then ψ has conjuncts $\mathsf{M}(Z_1, Z_2)$, …, $\mathsf{M}(Z_n, Z_1)$. If G had no reversal, then ψ would also have as conjuncts $\neg\mathsf{M}(Z_2, Z_1)$, …, $\neg\mathsf{M}(Z_1, Z_n)$. And using the logic, we would see that $\vdash \neg\psi$; that is, ψ would be inconsistent. We conclude from this contradiction that indeed every cycle in G has a reversal.

By Theorem 2.2, G is a majority digraph. This gives finite sets A_X for $X \in G$ with the property that for $X \neq Y$,

$$X \to Y \quad \text{iff} \quad |A_X \cap A_Y| > \frac{1}{2}|A_X| \ . \tag{4.1}$$

and hence we get a model: let $U = \bigcup_X A_X$, and let $[\![X]\!] = A_X$ when $X \in G$, and $[\![X]\!] = \emptyset$ when $X \notin G$.

We claim that $\mathcal{U} \models \psi$. For a conjunct of ψ of the form $\mathsf{M}(X,Y)$, we argue as follows: the first axiom of the logic having to do with M implies that both X and Y belong to G. And then the construction arranged that $\mathcal{U} \models \mathsf{M}(X,Y)$.

Consider a conjunct $\neg\mathsf{M}(X,Y)$. If both X and Y belong to G, and if $X \neq Y$, then the construction arranged that $\mathcal{U} \models \neg\mathsf{M}(X,Y)$. If either X or Y is not in G, then $[\![X]\!] = \emptyset$ or $[\![Y]\!] = \emptyset$, and again we have $\mathcal{U} \models \neg\mathsf{M}(X,Y)$. If $X, Y \in G$ and $X = Y$, then $\mathsf{M}(X,Y)$ is a conjunct of ψ by definition of G, and we contradict the consistency of ψ.

This completes the proof. □

We conclude with a remark on the satisfiability problem for $\mathcal{L}(\mathsf{most})$. By this we mean the question of whether a given sentence φ of $\mathcal{L}(\mathsf{most})$ has a model $\mathcal{U}$ in our sense: a finite set U and sets $[\![X]\!]$ which make φ true according to the definition. Note that every model $\mathcal{U}$ also gives us a truth assignment to the atomic sentences $\mathsf{M}(X,Y)$ of $\mathcal{L}(\mathsf{most})$. It is convenient to regard these atomic sentences $\mathsf{M}(X,Y)$ as "variables" and construct propositional logic over them. When we do this, then every model gives a truth assignment to these "variables."

Proposition 4.2. *The satisfiability problem for $\mathcal{L}(\mathsf{most})$ is NP-complete.*

Proof. Given a sentence φ, one can guess an assignment α and verify that α both satisfies φ and also corresponds to a model in our sense. This last point boils down to taking α and making a digraph G_α the way we did in the proof of Theorem 4.1: the vertices in G_α are the variables X such that $\alpha(\mathsf{M}(X,X)) = \text{true}$, and $X \to Y$ in G_α iff $\alpha(\mathsf{M}(X,Y)) = \text{true}$. We can check in polynomial time that G_α has the property that every cycle has a reversal.

In the other direction, we reduce 3SAT to our problem. Suppose we are given a 3SAT instance over a set $\{x_1, \ldots, x_n\}$ of boolean variables. We are going to consider

$\mathcal{L}(\mathsf{most})$ formulated over a set of (twice as many) variables $X_1, \ldots, X_{2n}$. Translate via $x_i \mapsto \mathsf{M}(X_{2i-1}, X_{2i})$. For example, a clause like $x_1 \vee \neg x_2 \vee x_3$ translates to

$$\mathsf{M}(X_1, X_2) \vee \neg \mathsf{M}(X_3, X_4) \vee \mathsf{M}(X_5, X_6).$$

Translate a 3SAT instance α clause-by-clause in this way. We only need to check that the translation preserves satisfiability; the converse is obvious. If our original 3SAT instance were satisfiable, we take a satisfying assignment α and convert it to a digraph G_α just as in our last paragraph. The point is that the translation α^t arranges that all of the edges in G_α are of the form $X_{2i-1} \to X_{2i}$ for some i. The structure of G_α makes it trivially a majority digraph: when $X_{2i-1} \to X_{2i}$ is an edge of G_α, let $A_{X_{2i-1}}$ be a singleton $\{2i-1\}$, let $A_{X_{2i}}$ be this set $\{2i-1\}$ with two more points; in all other cases, we take disjoint singletons. A finite model corresponding to G_α satisfies α^t. □

5. Conclusion and Further Questions

We have shown that a digraph G with no one-way cycles is a proportionality α-digraph for all $\alpha \in (0,1)$. But we do not know the smallest size of the sets A_v or of their union, as a function of α and $|G|$.

One could also study digraphs which are representable by the "exactly α" condition. That is, given $\alpha \in (0,1)$, which digraphs G have the property that there are finite sets A_v corresponding to the vertices of G such that $u \to v$ in G if and only if $|A_u \cap A_v| = \alpha \cdot |A_u|$?

For our last variations, suppose that $\alpha < \beta$ and that we ask of a digraph G that there be finite sets A_v such that $u \to v$ in G if and only if $\alpha \cdot |A_u| < |A_u \cap A_v| < \beta \cdot |A_u|$. Let us call this condition (α, β)*-proportionality.* We do not know the exact characterization of the class of all (α, β)-proportional digraphs. One can show that if a digraph G has no one-way cycles, then it is (α, β)-proportional. This is a corollary to the proof of Theorem 3.3 by taking ε and δ sufficiently small, all the numbers involved in Theorem 3.3 will be so close to α that β is irrelevant. But the converse is false: it is not necessary that a digraph have no one-way cycles in order for it to be (α, β)-proportional. For example, take $\beta = .99$, $\alpha = .5$, and G to be the one-way cycle $u \to v \to w \to u$. This digraph is (α, β)-proportional: take $A_u = \{0,1,2,3,4,5\}$, $A_v = \{0,1,2,3,6,7,8,9\}$, and $A_w = \{0,1,2,6,7\}$. Thus (α, β)-proportionality is weaker than the property of having no one-way cycles. So we leave open the exact characterization.

Similarly, we would say that a digraph is $]\alpha, \beta[$*-proportional* if there are finite sets A_v such that $u \to v$ in G if and only if $|A_u \cap A_v| \leq \alpha \cdot |A_u|$ or $\beta \cdot |A_u| \leq |A_u \cap A_v|$. Then G is (α, β)-proportional if and only if its complement G^c is $]\alpha, \beta[$-proportional. So the two concepts would have complementary characterizations. Again, we ask for a characterization of $]\alpha, \beta[$-proportional digraphs.

There is much more to be done on the logic of "most", since the language $\mathcal{L}(\mathsf{most})$ of Section 4 was extremely limited: by adding interesting expressions to that language, one quickly arrives at questions which seem interesting both from the viewpoints of logic and of combinatorics. For a different contribution to this project, see [EM].

Acknowledgments

It is a pleasure to thank Ian Pratt-Hartmann for his interest in, and comments on, this paper. We also thank several anonymous reviewers.

References

EM. Jörg Endrullis and Lawrence S. Moss, Syllogistic Logic with "Most", in V. de Paiva et al (eds.) *Proceedings, Workshop on Logic, Language, Information and Computation (WoLLIC'15)*, 2015, 215–229.

U. Chloe Urbanski, personal communication, 2013.

Institute for Mathematics and its Applications, University of Minnesota, Minneapolis MN 55455 USA.
E-mail address: `tmlai@ima.umn.edu`

Vrije Universiteit Amsterdam, 1081 HV Amsterdam, The Netherlands, and Department of Mathematics, Indiana University, Bloomington IN 47405 USA
E-mail address: `j.endrullis@vu.nl`

Department of Mathematics, Indiana University, Bloomington IN 47405 USA
E-mail address: `lsm@cs.indiana.edu`